D-Branes and Defects in the Liouville field theory

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Cardy-Lewellen equations in diagonal models

Let us collect the standard stuff on the 2d CFT.

Two-dimensional conformal transformation

$$z \to f(z) \qquad \bar{z} \to \bar{f}(\bar{z}) \tag{1}$$

Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(n^3 - n)\delta_{n,-m} \tag{2}$$

$$[\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n} + \frac{c}{12}(n^3 - n)\delta_{n,-m} \tag{3}$$

$$[L_m, \bar{L}_n] = 0 \tag{4}$$

Highest weight state $|h\rangle$ is defined by the relations

$$L_0|h\rangle = h|h\rangle \qquad L_m|h\rangle = 0 \qquad m > 0 \tag{5}$$

Primary field-state correspondence

$$\Phi(0)|0\rangle = |h\rangle \tag{6}$$

Descendant fields:

$$L_{-n_{i_1}} \dots L_{-n_{i_k}}|h\rangle \tag{7}$$

The Hilbert space of the conformal field theory has the form:

$$\mathcal{H} = \oplus_{i,\bar{i}} R_i(c) \otimes R_{\bar{i}}(c) \tag{8}$$

$R_i(c)$ is the chiral algebra highest weight i representation. The character is defined

$$\chi_i(\tau) = \mathrm{Tr}_{R_i} q^{L_0 - c/24} \tag{9}$$

The characters have the following property

$$\chi_i\left(-\frac{1}{\tau}\right) = S_i^j \chi_j(\tau) \tag{10}$$

S_i^j is the matrix of the modular transformation.

Two-point function is

$$\langle \Phi_1(z_1,\bar{z}_1)\Phi_2(z_2,\bar{z}_2)\rangle = \frac{C}{(z_1-z_2)^{2h}(\bar{z}_1-\bar{z}_2)^{2\bar{h}}} \tag{11}$$

where $h_1 = h_2 = h$, $\bar{h}_1 = \bar{h}_2 = \bar{h}$

$$\langle \Phi_1(z_1,\bar{z}_1)\Phi_2(z_2,\bar{z}_2)\Phi_3(z_3,\bar{z}_3)\rangle = \tag{12}$$
$$\frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_3+h_1-h_2}} \times \frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1} \bar{z}_{13}^{\bar{h}_3+\bar{h}_1-\bar{h}_2}}$$

where $z_{ij} = z_i - z_j$.

Bulk OPE has the form

$$\Phi_{(i\bar{i})}(z_1,\bar{z}_1)\Phi_{(j\bar{j})}(z_2,\bar{z}_2) = \tag{13}$$
$$\sum_{k,\bar{k},a,\bar{a}} \frac{C_{(i\bar{i})(j\bar{j})a\bar{a}}^{(k\bar{k})}}{(z_1-z_2)^{\Delta_i+\Delta_j-\Delta_k}(\bar{z}_1-\bar{z}_2)^{\Delta_{\bar{i}}+\Delta_{\bar{j}}-\Delta_{\bar{k}}}} \Phi_{(k\bar{k})}(z_2,\bar{z}_2)$$
$$+\text{descendants}$$

where $a = 1 \ldots N_{ij}^k$. The vacuum representation is indexed by $i=0$, and i^* refers to the conjugate representation in a sense $N^0_{ii^*}=1$. It is important to note that in the case of the models with multiplicities structure constants carry also additional indices a and $\bar{a}$ to disentangle different channels of the fusion.

The Verlinde formula is

$$N^k_{ij} = \sum_l \frac{S_{il}S_{jl}S^*_{lk}}{S_{0l}} \tag{14}$$

By the usual BPZ arguments we have for 4-point correlation function $\langle \Phi_i \Phi_k \Phi_j \Phi_l \rangle$ in s channel

$$\sum_{p\bar{p}} \sum_{\rho\tau\bar{\rho}\bar{\tau}} C^{p\bar{p}}_{j\bar{j}l\bar{l}(\tau\bar{\tau})} C^{i\bar{i}}_{k\bar{k}p\bar{p}(\rho\bar{\rho})} \mathcal{F}^s_{p\rho\tau} \begin{bmatrix} k & j \\ i & l \end{bmatrix} \mathcal{F}^s_{\bar{p}\bar{\rho}\bar{\tau}} \begin{bmatrix} \bar{k} & \bar{j} \\ \bar{i} & \bar{l} \end{bmatrix} \tag{15}$$

and t channel

$$\sum_{q\bar{q}}\sum_{\mu\nu\bar{\mu}\bar{\nu}} C^{q\bar{q}}_{k\bar{k}j\bar{j}(\mu\bar{\mu})} C^{i\bar{i}}_{q\bar{q}l\bar{l}(\nu\bar{\nu})} \mathcal{F}^t_{q\nu\mu}\left[\begin{array}{cc} k & j \\ i & l \end{array}\right] \mathcal{F}^t_{\bar{q}\bar{\nu}\bar{\mu}}\left[\begin{array}{cc} \bar{k} & \bar{j} \\ \bar{i} & \bar{l} \end{array}\right] \tag{16}$$

where $\mathcal{F}^s_{p\rho\tau}\left[\begin{array}{cc} k & j \\ i & l \end{array}\right]$ and $\mathcal{F}^t_{q\nu\mu}\left[\begin{array}{cc} k & j \\ i & l \end{array}\right]$ s and t channels conformal blocks correspondingly. Conformal blocks as well carry additional indices $\rho = 1\ldots N^i_{kp}$, $\tau = 1\ldots N^p_{jl}$, $\mu = 1\ldots N^q_{kj}$ $\nu = 1\ldots N^i_{ql}$, and similar for the right barred indices, to disentangle different fusion channels. Conformal blocks in s and t channels are related by the fusing matrix

$$\mathcal{F}^s_{p\rho\tau}\left[\begin{array}{cc} k & j \\ i & l \end{array}\right] = \sum_q\sum_{\nu\mu} F_{p,q}\left[\begin{array}{cc} k & j \\ i & l \end{array}\right]^{\nu\mu}_{\rho\tau} \mathcal{F}^t_{q\nu\mu}\left[\begin{array}{cc} k & j \\ i & l \end{array}\right] \tag{17}$$

$$\sum_{\rho\tau\bar{\rho}\bar{\tau}p\bar{p}} C^{p\bar{p}}_{j\bar{j}l\bar{l}(\tau\bar{\tau})} C^{i\bar{i}}_{k\bar{k}p\bar{p}(\rho\bar{\rho})} \times \tag{18}$$

$$F_{p,q}\left[\begin{array}{cc} k & j \\ i & l \end{array}\right]_{\rho\tau}^{\nu\mu} F_{\bar{p},\bar{q}}\left[\begin{array}{cc} \bar{k} & \bar{j} \\ \bar{i} & \bar{l} \end{array}\right]_{\bar{\rho}\bar{\tau}}^{\bar{\nu}\bar{\mu}}$$
$$= C^{q\bar{q}}_{k\bar{k}j\bar{j}(\mu\bar{\mu})} C^{i\bar{i}}_{q\bar{q}l\bar{l}(\nu\bar{\nu})}$$

Using the relation

$$\sum_{\bar{q},\bar{\nu},\bar{\mu}} F_{\bar{p},\bar{q}^*}\left[\begin{array}{cc} \bar{k} & \bar{j} \\ \bar{i} & \bar{l} \end{array}\right]_{\bar{\rho}\bar{\tau}}^{\bar{\nu}\bar{\mu}} F_{\bar{q},s}\left[\begin{array}{cc} \bar{j} & \bar{l} \\ \bar{k}^* & \bar{i}^* \end{array}\right]_{\bar{\mu}\bar{\nu}}^{\gamma_1\gamma_2} = \delta_{\bar{p}s}\delta_{\bar{\rho}\gamma_1}\delta_{\bar{\tau}\gamma_2} \tag{19}$$

Eq. (18) can be written in the form:

$$\sum_p \sum_{\rho\tau} C^{p\bar{p}}_{j\bar{j}l\bar{l}(\tau\bar{\tau})} C^{i\bar{i}}_{k\bar{k}p\bar{p}(\rho\bar{\rho})} F_{p,q}\left[\begin{array}{cc} k & j \\ i & l \end{array}\right]_{\rho\tau}^{\nu\mu} = \tag{20}$$
$$\sum_{\bar{q},\bar{\mu},\bar{\nu}} C^{q\bar{q}}_{k\bar{k}j\bar{j}(\mu\bar{\mu})} C^{i\bar{i}}_{q\bar{q}l\bar{l}(\nu\bar{\nu})} F_{\bar{q}^*,\bar{p}}\left[\begin{array}{cc} \bar{j} & \bar{l} \\ \bar{k}^* & \bar{i}^* \end{array}\right]_{\bar{\mu}\bar{\nu}}^{\bar{\rho}\bar{\tau}}$$

For diagonal model

$$C^{p\bar{p}}_{k\bar{k}i\bar{i}(\rho\bar{\rho})} = C^{p}_{ki(\rho\bar{\rho})}\delta_{\bar{p}p^*}\delta_{\bar{k}k^*}\delta_{\bar{i}i^*} \tag{21}$$

Eq. (20) takes the form:

$$
\sum_{\rho\tau} C^{i}_{kp(\rho\bar{\rho})} C^{p}_{jl(\tau\bar{\tau})} F_{p,q} \left[\begin{array}{cc} k & j \\ i & l \end{array} \right]^{\nu\mu}_{\rho\tau} = \tag{22}
$$

$$
\sum_{\bar{\mu}\bar{\nu}} C^{q}_{kj(\mu\bar{\mu})} C^{i}_{ql(\nu\bar{\nu})} F_{q,p} \left[\begin{array}{cc} k^{*} & i \\ j & l^{*} \end{array} \right]^{\bar{\tau}\bar{\rho}}_{\bar{\mu}\bar{\nu}}
$$

Pentagon equation for fusing matrix

$$\sum_{s,\beta_2,t_2,t_3} F_{p_2,s}\begin{bmatrix} j & k \\ p_1 & b \end{bmatrix}^{\beta_2 t_3}_{\alpha_2\alpha_3} F_{p_1,l}\begin{bmatrix} i & s \\ a & b \end{bmatrix}^{\gamma_1 t_2}_{\alpha_1\beta_2} \times \tag{23}$$
$$F_{s,r}\begin{bmatrix} i & j \\ l & k \end{bmatrix}^{u_2 u_3}_{t_2 t_3} = \sum_{\beta_1} F_{p_1,r}\begin{bmatrix} i & j \\ a & p_2 \end{bmatrix}^{\beta_1 u_3}_{\alpha_1\alpha_2} F_{p_2,l}\begin{bmatrix} r & k \\ a & b \end{bmatrix}^{\gamma_1 u_2}_{\beta_1\alpha_3}$$

implies the following important relation:

$$\sum_{\rho,\tau} F_{0,i}\left[\begin{array}{cc} p & k \\ p & k^* \end{array}\right]_{00}^{\bar{\rho}\rho} F_{p,q}\left[\begin{array}{cc} k & j \\ i & l \end{array}\right]_{\rho\tau}^{\nu\mu} F_{0,p}\left[\begin{array}{cc} l & j \\ l & j^* \end{array}\right]_{00}^{\bar{\tau}\tau} = \tag{24}$$

$$\sum_{\bar{\mu},\bar{\nu}} F_{0,q}\left[\begin{array}{cc} j & k \\ j & k^* \end{array}\right]_{00}^{\bar{\mu}\mu} F_{q,p}\left[\begin{array}{cc} k^* & i \\ j & l^* \end{array}\right]_{\bar{\mu}\bar{\nu}}^{\bar{\tau}\bar{\rho}} F_{0,i}\left[\begin{array}{cc} q & l \\ q & l^* \end{array}\right]_{00}^{\bar{\nu}\nu}$$

Comparing (24) and (22) we see that (22) can be solved by an ansatz

$$C_{ij(\mu\bar{\mu})}^{p} = \frac{\eta_i \eta_j}{\eta_0 \eta_p} F_{0,p}\left[\begin{array}{cc} j & i \\ j & i^* \end{array}\right]_{00}^{\bar{\mu}\mu} \tag{25}$$

with arbitrary η_i. To find η_i we set $p = 0$

$$C_{ii^*}^{0} = \frac{\eta_i \eta_{i^*}}{\eta_0^2} F_i \tag{26}$$

where

$$F_i \equiv F_{0,0}\left[\begin{array}{cc} i & i^* \\ i & i \end{array}\right] \tag{27}$$

Using

$$C_{ii^*}^{0} = \frac{C_{ii^*}}{C_{00}} \tag{28}$$

where C_{ii^*} are two-point functions

$$\langle\Phi_i(z_1,\bar{z}_1)\Phi_{i^*}(z_2,\bar{z}_2)\rangle = \frac{C_{ii^*}}{(z_1-z_2)^{2h}(\bar{z}_1-\bar{z}_2)^{2\bar{h}}} \tag{29}$$

and that $F_0 = 1$ one can solve (26) settting

$$\eta_i = \sqrt{C_{ii^*}/F_i} \tag{30}$$

Using the relation

$$F_{0,i}\left[\begin{array}{cc} j & k \\ j & k^* \end{array}\right] F_{i,0}\left[\begin{array}{cc} k^* & k \\ j & j \end{array}\right] = \frac{F_j F_k}{F_i} \tag{31}$$

we can write (25) for the models without multiplicities in two forms

$$C_{ij}^p = \frac{\eta_i\eta_j}{\eta_0\eta_p} F_{0,p}\left[\begin{array}{cc} j & i \\ j & i^* \end{array}\right] \tag{32}$$

and

$$C_{ij}^p = \frac{\xi_i\xi_j}{\xi_0\xi_p} \frac{1}{F_{p,0}\left[\begin{array}{cc} j^* & j \\ i & i \end{array}\right]} \tag{33}$$

where η_i is defined in (30) and

$$\xi_i = \eta_i F_i = \sqrt{C_{ii^*} F_i} \tag{34}$$

Conformal boundaries

$$T_{zz} = T = \bar{T} = T_{\bar{z}\bar{z}} \tag{35}$$

The relation (25) enables as to solve the Cardy-Lewellen cluster equations for various D-branes and defects. The Cardy-Lewellen cluster condition for one-point functions in the presence of boundary

$$\langle \Phi_{(i\bar{i})}(z,\bar{z}) \rangle_{(\alpha)} = \frac{U^i_{(\alpha)} \delta_{i^* \bar{i}}}{(z-\bar{z})^{2\Delta_i}} \tag{36}$$

reads

$$\sum_{k,a,\bar{a}} C^{(k,k^*)}_{(ii^*)(jj^*)a\bar{a}} \tilde{U}^k_{(\alpha)} F_{k0} \left[\begin{array}{cc} i^* & i \\ j & j \end{array} \right]^{00}_{\bar{a}a} = \tilde{U}^i_{(\alpha)} \tilde{U}^j_{(\alpha)} \tag{37}$$

where

$$U^i_{(\alpha)} = \tilde{U}^i_{(\alpha)} e^{i\pi\Delta_i} \tag{38}$$

To receive (37) consider now two-point function $\langle \Phi_i(z_1,\bar{z}_1)\Phi_j(z_2,\bar{z}_2)\rangle_\alpha$ in the presence of boundary in two pictures. In the first picture one applies first bulk OPE

$$\begin{aligned} &\Phi_{(i\bar{i})}(z_1,\bar{z}_1)\Phi_{(j\bar{j})}(z_2,\bar{z}_2) = \\ &\sum_{k,\bar{k},a,\bar{a}} \frac{C^{(k\bar{k})}_{(i\bar{i})(j\bar{j})a\bar{a}}}{(z_1-z_2)^{\Delta_i+\Delta_j-\Delta_k}(\bar{z}_1-\bar{z}_2)^{\Delta_{\bar{i}}+\Delta_{\bar{j}}-\Delta_{\bar{k}}}} \Phi_{(k\bar{k})}(z_2,\bar{z}_2) \\ &+ \ldots \end{aligned} \tag{39}$$

and then evaluates one-point function resulting in:

$$\langle\Phi_{(i\bar{i})}(z_1,\bar{z}_1)\Phi_{(j\bar{j})}(z_2,\bar{z}_2)\rangle_\alpha = \sum_{k,a,\bar{a}} C^{(k,k^*)}_{(i\bar{i})(j\bar{j})a\bar{a}} U^k_\alpha \mathcal{F}^{ij\overline{ij}}_{ka\bar{a}} \tag{40}$$

where $\mathcal{F}^{ij\overline{ij}}_{ka\bar{a}}$ are conformal blocks.

In the second picture one first applies bulk-boundary OPE

$$\Phi_{(i\bar{i})}(z,\bar{z}) = \sum_{m,t,s} \frac{R^{(i\bar{i}),t}_{m,s,(\alpha)}}{(z-\bar{z})^{\Delta_i+\Delta_{\bar{i}}-\Delta_m}} \psi^{\alpha\alpha,s}_m + \ldots \tag{41}$$

where $t = 1,\ldots N^m_{i\bar{i}}$, and index s counts different boundary fields and runs $s = 1,\ldots n^m_{\alpha\alpha}$, where $n^m_{\alpha\alpha}$ coefficient of character χ_m in the annulus partition function between brane α with itself, and then evaluates two-point function of boundary fields resulting in

$$\langle\Phi_{(i\bar{i})}(z_1,\bar{z}_1)\Phi_{(j\bar{j})}(z_2,\bar{z}_2)\rangle_\alpha = \sum_{m,t_1,t_2,s_1,s_2} R^{(i\bar{i}),t_1}_{m,s_1(\alpha)} R^{(j\bar{j}),t_2}_{m^*,s_2(\alpha)} \mathcal{F}^{i\bar{i}j\bar{j}}_{mt_1t_2} c^{\alpha,s_1,s_2}_m \tag{42}$$

where

$$\langle\psi^{\alpha\alpha,s_1}_m(x_1)\psi^{\alpha\alpha,s_2}_n(x_2)\rangle = \frac{c^{\alpha,s_1,s_2}_m \delta_{mn^*}}{|x_2-x_1|^{2\Delta_m}} \tag{43}$$

Using braiding relations between chiral blocks

$$\mathcal{F}^{ij\overline{i}\overline{j}}_{ka\bar{a}} = \sum_m B^{(+)}_{k^*m^*}\left[\begin{array}{cc} j & \bar{i} \\ i^* & \bar{j}\end{array}\right]^{t_1t_2}_{a\bar{a}} \mathcal{F}^{i\bar{i}j\bar{j}}_{mt_1t_2} \tag{44}$$

one derives:

$$\sum_{k,a,\bar{a}} C^{(k,k^*)}_{(i\bar{i})(j\bar{j})a\bar{a}} U^k_\alpha B^{(+)}_{k^*m^*}\left[\begin{array}{cc} j & \bar{i} \\ i^* & \bar{j}\end{array}\right]^{t_1t_2}_{a\bar{a}} = \sum_{s_1,s_2} R^{(i\bar{i}),t_1}_{m,s_1,(\alpha)} R^{(j\bar{j}),t_2}_{m^*,s_2,(\alpha)} c^{\alpha,s_1,s_2}_m \tag{45}$$

Putting $m=0$ one obtains:

$$\sum_{k,a,\bar{a}} C^{(k,k^*)}_{(ii^*)(jj^*)a\bar{a}} U^k_\alpha B^{(+)}_{k^*0} \left[\begin{array}{cc} j & i^* \\ i^* & j^* \end{array} \right]^{11}_{a\bar{a}} = U^i_{(\alpha)} U^j_{(\alpha)} \tag{46}$$

where we took into account that $R^{i\bar{i}}_{0(\alpha)} = U^i_\alpha \delta_{i^*\bar{i}}$.

We should note that traditionally used reflection amplitudes differ by the phase

$$U^i_{(\alpha)} = \tilde{U}^i_{(\alpha)} e^{i\pi\Delta_i} \tag{47}$$

They have the advantage, that related to boundary states coefficients without phase factor:

$$\tilde{U}^i_{(\alpha)} = \frac{B^i_\alpha}{B^0_\alpha} \tag{48}$$

Recalling relation between braiding and fusion matrices:

$$B^{(+)}_{pq} \left[\begin{array}{cc} i & j \\ k & l \end{array} \right]^{cd}_{ab} = e^{i\pi(\Delta_k+\Delta_l-\Delta_p-\Delta_q)} F_{pq} \left[\begin{array}{cc} i & l \\ k & j \end{array} \right]^{cd}_{ab} \tag{49}$$

and symmetry properties of fusion matrix

$$F_{pq}\left[\begin{array}{cc} k & j \\ i & l \end{array}\right]_{ab}^{cd} = F_{p^*q^*}\left[\begin{array}{cc} l & i^* \\ j^* & k \end{array}\right]_{ab}^{cd} \qquad (50)$$

we receive that $\tilde{U}^i_{(\alpha)}$ obey the equation:

$$\sum_{k,a,\bar{a}} C^{(k,k^*)}_{(ii^*)(jj^*)a\bar{a}} \tilde{U}^k_\alpha F_{k0}\left[\begin{array}{cc} i^* & i \\ j & j \end{array}\right]_{a\bar{a}}^{11} = \tilde{U}^i_{(\alpha)}\tilde{U}^j_{(\alpha)} \qquad (51)$$

Putting (25) in (37), and performing the sums by a and $\bar{a}$ using the relations

$$F_i F_{0,i^*}\left[\begin{array}{cc} j & k \\ j & k^* \end{array}\right]_{00}^{\nu u_3} = F_{k^*} F_{0,k^*}\left[\begin{array}{cc} i & j \\ i & j^* \end{array}\right]_{00}^{\nu u_3} \qquad (52)$$

$$\sum_{t_2} F_{0,l}\left[\begin{array}{cc} a & b^* \\ a & b \end{array}\right]_{00}^{\gamma_1 t_2} F_{b^*,0}\left[\begin{array}{cc} a & a^* \\ l & l \end{array}\right]_{t_2\alpha_3}^{00} = \qquad (53)$$

$$F_{0,0}\left[\begin{array}{cc} a & a^* \\ a & a \end{array}\right]_{00}^{00} \delta_{\gamma_1,\alpha_3} \equiv F_a \delta_{\gamma_1,\alpha_3}$$

we obtain

$$\sum_k U^k N_{ij}^k \frac{\xi_i \xi_j}{\xi_0 \xi_k} = U^i U^j \tag{54}$$

where N_{ij}^k are the fusion coefficients. Defining

$$U^k = \Psi^k \frac{\xi_k}{\xi_0} \tag{55}$$

one can write (54) in the form:

$$\sum_k \Psi^k N_{ij}^k = \Psi^i \Psi^j \tag{56}$$

Topological defects

Conformal defects are required to satisfy

$$T^{(1)} - \bar{T}^{(1)} = T^{(2)} - \bar{T}^{(2)} \tag{57}$$

Topological defect lines are defined by the conditions:

$$T^{(1)} = T^{(2)} \tag{58}$$

$$\bar{T}^{(1)} = \bar{T}^{(2)} \tag{59}$$

After modular transformation these defects are given by operators X, satisfying relations:

$$[L_n, X] = [\bar{L}_n, X] = 0 \tag{60}$$

As consequence of (60) X is a sum of projectors

$$X = \sum_{i,\bar{i}} \mathcal{D}^{(i,\bar{i})} P^{(i,\bar{i})} \tag{61}$$

where

$$P^{(i,\bar{i})} = \sum_{N,\bar{N}} (|i, N\rangle \otimes |\bar{i}, \bar{N}\rangle)(\langle i, N| \otimes \langle \bar{i}, \bar{N}|) \tag{62}$$

Two-point functions in the presence of defect

$$D^i = \frac{\mathcal{D}^{(i,\bar{i})}}{\mathcal{D}^0} \tag{63}$$

$$\langle \Phi_{ii^*}(z_1,\bar z_1) X \Phi_{i^*i}(z_2,\bar z_2)\rangle = \frac{D^i}{(z_1-z_2)^{2\Delta_i}(\bar z_1-\bar z_2)^{2\Delta_i}} \tag{64}$$

satisfy the cluster condition

$$\sum_k (C^1_{k^*k} D^{(k\bar k)}) C^k_{ij,a\bar a} C^{k^*}_{i^*j^*,c\bar c} \times \tag{65}$$
$$F_{k0}\left[\begin{array}{cc} j^* & j \\ i & i \end{array}\right]^{11}_{ac} F_{\bar k 0}\left[\begin{array}{cc} \bar j^* & \bar j \\ \bar i & \bar i \end{array}\right]^{11}_{\bar a\bar c} =$$
$$(C^1_{i^*i} D^{(i\bar i)})(C^1_{(j^*j} D^{(j\bar j)})$$

Derive the cluster condition for defects.

Here we should consider the two-point functions

$$\langle \Phi_{i^*}(z_1,\bar z_1) X \Phi_{i}(z_2,\bar z_2)\rangle = \frac{D^{(i,\bar i)}}{(z_1-z_2)^{2\Delta_i}(\bar z_1-\bar z_2)^{2\Delta_{\bar i}}} \tag{66}$$

$$D^i = \frac{\mathcal{D}^{(i,\bar i)}}{\mathcal{D}^0} \tag{67}$$

One can write for the following four-point function with the defects insertion in the first picture:

$$\langle \Phi_{j^*}(z_1,\bar{z}_1)\,\Phi_{i^*}(z_2,\bar{z}_2)\,X\,\Phi_i(z_3,\bar{z}_3)\,\Phi_j(z_4,\bar{z}_4)\,X^\dagger\rangle = \tag{68}$$
$$\sum_k C^1_{j^*j} C^k_{ij,a\bar{a}} C^j_{i^*k,c\bar{c}} D^{(k,\bar{k})} \mathcal{F}^{j^*i^*ij}_{kac} \mathcal{F}^{\overline{j}^*\overline{i}^*\overline{ij}}_{\overline{k}\overline{a}\overline{c}}$$

Here we denoted $C^k_{ij} \equiv C^{(kk^*)}_{(ii^*)(jj^*)}$ as before.

Using relations:

$$C^j_{i^*k,c\bar{c}} = C^j_{ki^*,c\bar{c}} \tag{69}$$

and

$$C^j_{ki^*,c\bar{c}} C^1_{j^*j} = C^{k^*}_{i^*j^*,c\bar{c}} C^1_{k^*k} \tag{70}$$

we can write for the second line of (68)

$$\sum_k C^k_{ij,a\bar{a}} C^{k^*}_{i^*j^*,c\bar{c}} C^1_{k^*k} D^{(k,\bar{k})} \mathcal{F}^{j^*i^*ij}_{kac} \mathcal{F}^{\overline{j}^*\overline{i}^*\overline{ij}}_{\overline{k}\overline{a}\overline{c}} \tag{71}$$

In the second picture one has:

$$\langle \Phi_{i^*}(z_2,\bar{z}_2) X \Phi_i(z_3,\bar{z}_3)\Phi_j(z_4,\bar{z}_4) X^\dagger \Phi_{j^*}(z_1,\bar{z}_1)\rangle = \tag{72}$$
$$C^1_{i^*i} C^1_{j^*j} D^{(i,\bar{i})} D^{(j,\bar{j})} \mathcal{F}_0^{i^*ijj^*} \mathcal{F}_0^{\overline{i}^*\overline{i}\overline{j}\overline{j}^*} + \ldots$$

Relating conformal blocks via the fusion matrix

brings to (65).

Inserting again (25) in (65) we obtain

$$\sum_k (C^1_{k^*k} D^{(k\bar{k})}) N^k_{ij} \left(\frac{\xi_i\xi_j}{\xi_0\xi_k}\right)^2 = (C^1_{i^*i} D^{(i\bar{i})})(C^1_{(j^*j} D^{(j\bar{j})}) \tag{73}$$

$$D^k = \Psi^k \left(\frac{\xi_k}{\xi_0}\right)^2 \frac{C_{00}}{C_{kk^*}} = \Psi^k F_k \tag{74}$$

with Ψ^k satisfying (56). In rational conformal field theory one has also the relation

$$F_k = \frac{S_{00}}{S_{0k}} \qquad (75)$$

In RCFT two-points functions can be normalized to 1. Therefore in RCFT $\xi_k = \frac{\sqrt{S_{00}}}{\sqrt{S_{0k}}}$. Eq. (56) is solved by

$$\Psi_a^k = \frac{S_{ak}}{S_{0a}} \qquad (76)$$

Taking also into account the relation between one-point functions U^k and coefficients of the boundary state B^k

$$U^k = \frac{B^k}{B^0} \qquad (77)$$

we obtain the formulae for the Cardy states

$$B_a^k = \frac{S_{ak}}{\sqrt{S_{0k}}} \qquad (78)$$

and defects

$$\mathcal{D}_a^k = \frac{S_{ak}}{S_{0k}} \tag{79}$$

In the case of non-rational theories one may have also continuous family of the boundary states, which can be obtained in the following way. Assume we consider the Cardy-Lewellen Eq. (54) for j being a fixed degenerate state and i is a generic state. One can treat in this case U^j as a constant parameter A characterizing a boundary condition . Setting $U^j = A$ one gets linear equation

$$\sum_k \Lambda^k N_{ij}^k = \Lambda^i A \frac{\xi_0}{\xi_j} \tag{80}$$

where

$$U^k = \Lambda^k \xi^k \tag{81}$$

Correspondingly the continuous family of defects is given by the following functions

$$D^k = \Lambda^k F_k \tag{82}$$

Liouville field theory

Let us review basic facts on the Liouville field theory . Liouville field theory is defined on a two-dimensional surface with metric g_{ab} by the local Lagrangian density

$$\mathcal{L} = \frac{1}{4\pi} g_{ab} \partial_a \varphi \partial_b \varphi + \mu e^{2b\varphi} + \frac{Q}{4\pi} R\varphi \tag{83}$$

where R is associated curvature. This theory is conformal invariant if the coupling constant b is related with the background charge Q as

$$Q = b + \frac{1}{b} \tag{84}$$

The symmetry algebra of this conformal field theory is the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c_L}{12}(n^3 - n)\delta_{n,-m} \quad (85)$$

with the central charge

$$c_L = 1 + 6Q^2 \quad (86)$$

Primary fields V_α in this theory, which are associated with exponential fields $e^{2\alpha\varphi}$, have conformal dimensions

$$\Delta_\alpha = \alpha(Q - \alpha) \quad (87)$$

The fields V_α and $V_{Q-\alpha}$ have the same conformal dimensions and represent the same primary field, i.e. they are proportional to each other:

$$V_\alpha = S(\alpha)V_{Q-\alpha} \quad (88)$$

with the function

$$S(\alpha) = \frac{\left(\pi\mu\gamma(b^2)\right)^{\frac{(Q-2\alpha)}{b}}}{b^2} \frac{\Gamma(1-b(Q-2\alpha))\Gamma(-b^{-1}(Q-2\alpha))}{\Gamma(b(Q-2\alpha))\Gamma(1+b^{-1}(Q-2\alpha))} \tag{89}$$

Two-point functions of Liouville theory are given by the reflection function (89):

$$\langle V_\alpha(z_1,\bar{z}_1)V_\alpha(z_2,\bar{z}_2)\rangle = \frac{S(\alpha)}{(z_1-z_2)^{2\Delta_\alpha}(\bar{z}_1-\bar{z}_2)^{2\Delta_\alpha}} \tag{90}$$

The spectrum of the Liouville theory is to be of the following form

$$\mathcal{H} = \int_0^\infty dP\ R_{\frac{Q}{2}+iP} \otimes R_{\frac{Q}{2}+iP} \tag{91}$$

where R_α is the highest weight representation with respect to Virasoro alegbra. Characters of the representations $R_{\frac{Q}{2}+iP}$ are

$$\chi_P(\tau) = \frac{q^{P^2}}{\eta(\tau)} \tag{92}$$

where

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \tag{93}$$

Modular transformation of (92) is well-known:

$$\chi_P(-\frac{1}{\tau}) = \sqrt{2} \int \chi_{P'}(\tau) e^{4i\pi PP'} dP' \tag{94}$$

Degenerate representations appear at $\alpha_{m,n} = \frac{1-m}{2b} + \frac{1-n}{2}b$ and have conformal dimensions

$$\Delta_{m,n} = Q^2/4 - (m/b + nb)^2/4 \tag{95}$$

where m, n are positive integers. At general b there is only one null-vector at the level mn. Hence the degenerate character reads:

$$\chi_{m,n}(\tau) = \frac{q^{-(m/b+nb)^2} - q^{-(m/b-nb)^2}}{\eta(\tau)} \tag{96}$$

Modular transformation of (96) is worked out to be

$$\chi_{m,n}(-\frac{1}{\tau}) = 2\sqrt{2}\int \chi_P(\tau)\,\mathrm{sinh}(2\pi mP/b)\,\mathrm{sinh}(2\pi nbP)dP \tag{97}$$

Given that the identity field is specified by $(m,n) = (1,1)$ one finds the vacuum component of the matrix of modular transformation:

$$S_{0\alpha} = -i2\sqrt{2}\,\mathrm{sin}\,\pi/b(2\alpha-Q)\,\mathrm{sin}\,\pi b(2\alpha-Q) \tag{98}$$

We have all the necessary ingredients to compute classifying algebra: two-point function $S(\alpha)$ and vacuum component of the matrix of modular transformation. Before to continue let us recall that both of them can be conveniently written using ZZ function:

$$W(\alpha) = -\frac{2^{3/4}e^{3i\pi/2}(\pi\mu\gamma(b^2))^{-\frac{(Q-2\alpha)}{2b}}\pi(Q-2\alpha)}{\Gamma(1-b(Q-2\alpha))\Gamma(1-b^{-1}(Q-2\alpha))} \tag{99}$$

It can be easily shown that

$$\frac{W(Q-\alpha)}{W(\alpha)} = S(\alpha) \tag{100}$$

and

$$W(Q-\alpha)W(\alpha) = S_{0\alpha} \tag{101}$$

Recalling (75) F_α takes the form:

$$F_\alpha = \frac{S_{00}}{W(Q-\alpha)W(\alpha)} \tag{102}$$

Combining (100) and (102) we obtain coefficients ξ_α for the Liouville field theory:

$$\xi^L_\alpha = \sqrt{S(\alpha)F(\alpha)} = \frac{\sqrt{S_{00}}}{W(\alpha)} \tag{103}$$

Eq. (33) implies:

$$C^{\alpha_3}_{\alpha_1,\alpha_2} F_{\alpha_3,0} \left[\begin{array}{cc} \alpha_1 & \alpha_1 \\ \alpha_2 & \alpha_2 \end{array} \right] = W(0) \frac{W(\alpha_3)}{W(\alpha_1)W(\alpha_2)} \tag{104}$$

Recalling the relation between three-point function and OPE structure constant

$$C^{\alpha_3}_{\alpha_1,\alpha_2} = C(\alpha_1, \alpha_2, Q - \alpha_3) \tag{105}$$

we obtain also the corresponding relation for three-point function

$$C(\alpha_1, \alpha_2, \alpha_3) F_{\alpha_3,0} \left[\begin{array}{cc} \alpha_1 & \alpha_1 \\ \alpha_2 & \alpha_2 \end{array} \right] = W(0) \frac{W(Q - \alpha_3)}{W(\alpha_1)W(\alpha_2)} \tag{106}$$

Note that (106) evidently satisfies the reflection property

$$C(\alpha_3, \alpha_2, \alpha_1) = S(\alpha_3) C(Q - \alpha_3, \alpha_2, \alpha_1) \tag{107}$$

since the fusing matrix is invariant under the inversions $\alpha \to Q-\alpha$.

With $j=-\frac{b}{2}$, $i=\alpha$, and $k=\alpha \pm b/2$, the equation (56) takes the form

$$\Psi(\alpha)\Psi(-b/2)=\Psi(\alpha-b/2)+\Psi(\alpha+b/2) \quad (108)$$

The solution of the equations (108) is

$$\Psi_{m,n}(\alpha)=\frac{\sin(\pi m b^{-1}(2\alpha-Q))\sin(\pi n b(2\alpha-Q))}{\sin(\pi m b^{-1}Q)\sin(\pi n b Q)}=\frac{S_{m,n\,\alpha}}{S_{m,n\,0}} \quad (109)$$

Therefore we have discrete family of branes (ZZ branes)

$$B_{m,n}(\alpha)=\frac{S_{m,n\,\alpha}}{W(\alpha)} \quad (110)$$

and defects

$$\mathcal{D}_{m,n}(\alpha)=\frac{S_{m,n\,\alpha}}{S_{0\alpha}} \quad (111)$$

Treating $\Psi(-b/2)$ as a continuous parameter A characterizing the defects one gets linear equation

$$A\Lambda(\alpha) = \Lambda(\alpha - b/2) + \Lambda(\alpha + b/2) \qquad (112)$$

Solution of (112) is

$$\Lambda_s(\alpha) = S_{s\alpha} = 2^{1/2}\cosh(2\pi s(2\alpha - Q)) \qquad (113)$$

with

$$2\cosh 2\pi bs = A \qquad (114)$$

and we have continuous family of branes (FZZ branes)

$$B_s(\alpha) = \frac{S_{s\alpha}}{W(\alpha)} \qquad (115)$$

and defects

$$\mathcal{D}_s(\alpha) = \frac{S_{s\alpha}}{S_{0\alpha}} \tag{116}$$