Elements of Geometric Quantization
&
Applications to Fields and Fluids

V. P. Nair

City College of the CUNY

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PLAN

Quantum theory is defined as a unitary irreducible representation of the algebra of observables.

Geometric quantization gives a way to realize this, elucidating the role of the geometry and topology of the phase space.

- Classical phase space dynamics
- Pre-quantum Hilbert space, operators, polarization
- Role of topology: $\mathcal{H}^1(M, \mathbb{R}),\ \mathcal{H}^2(M, \mathbb{R})$
- Quantizing S^2 and G/H
- Chern-Simons theory (and WZW theory)
- θ-vacua in gauge theories
- Statistics of holes in the fractional quantum Hall effect
- Fluid dynamics (Group theoretic approach and anomalies)

THE SYMPLECTIC STRUCTURE

Phase space = A smooth even dimensional manifold M endowed with a symplectic structure Ω

- Ω is a differential 2-form on M which is closed and nondegenerate.
  - Closed: $d\Omega = 0$
  - Nondegenerate: For any vector field ξ on M, $i_\xi \Omega = 0 \ \Rightarrow \xi = 0$

$$\Omega \ = \ \tfrac{1}{2}\ \Omega_{\mu\nu}\ dq^\mu \wedge dq^\nu$$

- The condition $d\Omega = 0$ becomes

$$\begin{aligned} d\Omega &= \frac{\partial \Omega_{\mu\nu}}{\partial q^\alpha} dq^\alpha \wedge dq^\mu \wedge dq^\nu \\ &= \frac{1}{3}\left[\frac{\partial \Omega_{\mu\nu}}{\partial q^\alpha} + \frac{\partial \Omega_{\alpha\mu}}{\partial q^\nu} + \frac{\partial \Omega_{\nu\alpha}}{\partial q^\mu}\right] dq^\alpha \wedge dq^\mu \wedge dq^\nu \\ &= 0 \end{aligned}$$

- Interior contraction with $\xi = \xi^\mu (\partial/\partial q^\mu)$ is

$$i_\xi \Omega = \xi^\mu \Omega_{\mu\nu} dq^\nu$$

$i_\xi \Omega = 0 \Rightarrow \xi = 0$ $\equiv$ $\xi^\mu \Omega_{\mu\nu} = 0 \Rightarrow \xi^\mu = 0$; $\Longleftrightarrow$ Ω is nondegenerate as a matrix

THE SYMPLECTIC STRUCTURE (*cont'd.*)

- Inverse of Ω can be defined by

$$\Omega_{\mu\nu}\,\Omega^{\nu\alpha} = \delta_\mu^{\ \alpha}$$

  (If Ω has zero modes, one has gauge symmetries.)

- Since $d\Omega = 0$, we can write

$$\Omega = d\mathcal{A} \qquad \Omega_{\mu\nu} = \frac{\partial}{\partial q^\mu}\mathcal{A}_\nu - \frac{\partial}{\partial q^\nu}\mathcal{A}_\mu$$

- What are the qualifications to this statement?
  - If there are noncontractible 2d-surfaces Σ such that

$$\int_\Sigma \Omega \neq 0$$

    then $\mathcal{A}$ cannot exist globally. (Equivalent to $\mathcal{H}^2(M) \neq 0$; e.g. CS, WZW theories)
  - Even if $\mathcal{H}^2(M) = 0$, one can have inequivalent $\mathcal{A}$'s. For example, $\mathcal{A}$ and $\mathcal{A} + A$ give same Ω if $dA = 0$.
    - Evidently $A = d\Lambda$ is one possibility (Canonical transformations)
    - One can have $A \neq d\Lambda$ with $dA = 0 \Longleftrightarrow \mathcal{H}^1(M) \neq 0$ (e.g. θ-vacua)

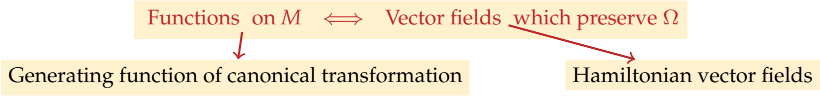
CANONICAL TRANSFORMATIONS

- Transformations of (phase space) coordinates which preserve Ω are canonical transformations.
- For infinitesimal transformations, $q^\mu \to q^\mu + \xi^\mu$, change in Ω is

$$\begin{aligned} \delta\Omega &= \left[\tfrac{1}{2}\Omega_{\mu\nu}(q+\xi)d(q^\mu+\xi^\mu)\wedge d(q^\nu+\xi^\nu) - \tfrac{1}{2}\Omega_{\mu\nu}(q)dq^\mu \wedge dq^\nu\right] \equiv L_\xi\Omega \\ &= d(i_\xi\Omega) + i_\xi d\Omega = d(i_\xi\Omega) \\ &= 0 \end{aligned}$$

The solution is $i_\xi\Omega = -df$ (if $\mathcal{H}^1(M) = 0$).

- Conversely, for any function f, one can define $\xi^\mu = \Omega^{\mu\nu}\partial_\nu f \Longrightarrow L_\xi\Omega = 0$.
- This leads to



Functions on M $\Longleftrightarrow$ Vector fields which preserve Ω

Generating function of canonical transformation

Hamiltonian vector fields

CANONICAL TRANSFORMATIONS (*cont'd.*)

- If ξ and η preserve Ω, so does their Lie commutator

$$[\xi, \eta]^\mu = \xi^\nu \partial_\nu \eta^\mu - \eta^\nu \partial_\nu \xi^\mu$$

- If $\xi \leftrightarrow f$ and $\eta \leftrightarrow g$, then there is a function corresponding to $[\xi, \eta]$; this is called the Poisson bracket $-\{f, g\}$ and is defined by

$$\{f, g\} = i_\xi i_\eta \Omega = \eta^\mu \xi^\nu \Omega_{\mu\nu} = -i_\xi dg = i_\eta df = \Omega^{\mu\nu} \partial_\mu f \partial_\nu g$$

- The Poisson bracket obeys

$$\begin{aligned} \{f, g\} &= -\{g, f\} \\ \{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} &= 0 \end{aligned}$$

- Poisson brackets are important because the change in a function on phase space due to a canonical transformation is

$$\delta F = \xi^\mu \partial_\mu F = \{F, f\}$$

CANONICAL TRANSFORMATIONS (*cont'd.*)

- The change in the canonical 1-form is given by

$$\delta\mathcal{A} = L_\xi \mathcal{A} = d(i_\xi \mathcal{A} - f) = d\Lambda$$

- Classical dynamics is given by

$$\Omega_{\mu\nu} \frac{\partial q^\nu}{\partial t} = \frac{\partial H}{\partial q^\mu}$$

- This can be obtained from an action

$$\mathcal{S} = \int_{t_i}^{t_f} dt \, \left(\mathcal{A}_\mu \frac{dq^\mu}{dt} \; - \; H \right)$$

- Variation of the action gives

$$\delta\mathcal{S} = i_\xi \mathcal{A}(t_f) - i_\xi \mathcal{A}(t_i) + \int dt \, \left(\Omega_{\mu\nu} \frac{dq^\nu}{dt} - \frac{\partial H}{\partial q^\mu} \right) \xi^\mu$$

- Given the action, the boundary term in its variation can be used to identify $\mathcal{A}$ and, hence, Ω.

CANONICAL TRANSFORMATIONS (*cont'd.*)

- As an example, consider the usual scalar field theory with

$$\mathcal{S} = \int d^4x \left[\frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}(\nabla\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \alpha\varphi^4 \right]$$

- The variation of the action leads, upon time-integration, to the boundary term

$$\delta\mathcal{S} = \int d^3x\, \dot{\varphi}\, \delta\varphi \Big]_{t_i}^{t_f} + \int d^4x\, [\cdots] \implies \mathcal{A} = \int d^3x\, \dot{\varphi}\, \delta\varphi$$

- A less obvious case is the quantization in lightcone coordinates. Define

$$u = \frac{1}{\sqrt{2}}(t+z), \qquad v = \frac{1}{\sqrt{2}}(t-z)$$

- In this case

$$\mathcal{S} = \int du\, dv\, d^2x^T\, [\partial_u\varphi\partial_v\varphi - \cdots] \implies \mathcal{A} = \int dv\, d^2x^T\, \partial_v\varphi\, \delta\varphi$$

QUANTIZATION

Quantum Theory = Unitary Irreducible Representation of the Algebra of Observables

- The problem of quantization is: How do we realize this explicitly?
  - Canonical transformations $\Longleftrightarrow$ Unitary transformations
  - (Poisson bracket) classical algebra of observables $\Longleftrightarrow$ Commutator algebra of operators
  - Ensure irreducibility
- Geometric quantization provides a way to do this

STRATEGY:

1. Define pre-quantum wave functions and pre-quantum operators
2. Impose a polarization to achieve irreducibility

QUANTIZATION (*cont'd.*)

- Since canonical transformations are $\mathcal{A} \to \mathcal{A} + d\Lambda$, we consider wave functions to have the property

$$\Psi(q) \to e^{i\Lambda}\, \Psi(q), \qquad \mathcal{A} \to \mathcal{A} + d\Lambda$$

- Ψ depends on all phase space coordinates. They are analogous to fields coupled to a $U(1)$ gauge field $\mathcal{A}$. (They are sections of a line bundle on M with curvature Ω.)
- The Ψ's are pre-quantum wave functions and form a (pre-quantum) Hilbert space with the inner product

$$(1|2) \;=\; \int d\sigma(M)\; \Psi_1^*\; \Psi_2$$

$d\sigma(M) = \Omega \wedge \Omega \cdots \wedge \Omega \sim \det(\Omega)\, d^{2n}q$.

- How does Ψ change under $q^\mu \to q^\mu + \xi^\mu$? Under such a change, $\mathcal{A} \to \mathcal{A} + i_\xi \mathcal{A} - f$, so that

$$\begin{aligned} \delta\Psi &= \xi^\mu \partial_\mu \Psi \;-\; i(\, i_\xi \mathcal{A} - f)\Psi \\ &= \xi^\mu\, (\partial_\mu - i\mathcal{A}_\mu)\, \Psi + i\, f\; \Psi \;=\; (\xi^\mu \mathcal{D}_\mu + i\, f)\;\; \Psi \end{aligned}$$

The first term gives change of Ψ as a function, the second compensates for the change of $\mathcal{A}$.

QUANTIZATION (*cont'd.*)

- Define the pre-quantum operator corresponding to f as

$$\mathcal{P}(f) = -i(\xi \cdot \mathcal{D} + if)$$

- In terms of Hamiltonian vector fields, $f \leftrightarrow \xi$, $g \leftrightarrow \eta$, $\{f, g\} \leftrightarrow -[\xi, \eta]$; this gives

$$\begin{aligned}
[\mathcal{P}(f), \mathcal{P}(g)] &= [-i\xi \cdot \mathcal{D} + f, -i\eta \cdot \mathcal{D} + g] \\
&= -[\xi^\mu \mathcal{D}_\mu, \eta^\nu \mathcal{D}_\nu] - i\xi^\mu [\mathcal{D}_\mu, g] + i\eta^\mu [\mathcal{D}_\mu, f] \\
&= i\xi^\mu \eta^\nu \Omega_{\mu\nu} - (\xi^\mu \partial_\mu \eta^\nu)\mathcal{D}_\nu + (\eta^\mu \partial_\mu \xi^\nu)\mathcal{D}_\nu - i\xi^\mu \partial_\mu g + i\eta^\mu \partial_\mu f \\
&= i\left(-\xi^\mu \eta^\nu \Omega_{\mu\nu} + i[\xi, \eta] \cdot \mathcal{D}\right) \\
&= i\left(-i\,(i_{[\eta,\xi]}\mathcal{D}) + \{f, g\}\right) \\
&= i\,\mathcal{P}(\{f, g\})
\end{aligned}$$

- The pre-quantum operators form a representation of the Poisson bracket algebra of functions on the phase space, with $[A, B] \sim i\{A, B\}$.

QUANTIZATION (*cont'd.*)

- We get a representation, but this is reducible in general, since Ψ depends on all phase space variables.
- Illustrate by example: Point-particle in one space dimension

$$\Omega = dp \wedge dx, \qquad \mathcal{A} = p\, dx$$

- Hamiltonian vector fields and pre-quantum operators for q and p are

$$x \longleftrightarrow -\frac{\partial}{\partial p}, \qquad p \longleftrightarrow \frac{\partial}{\partial x}$$

$$\mathcal{P}(x) = i\frac{\partial}{\partial p} + x, \qquad \mathcal{P}(p) = -i\left(\frac{\partial}{\partial x} - ip\right) + p = -i\frac{\partial}{\partial x}$$

  $[\mathcal{P}(x), \mathcal{P}(p)] = i$, so that we have a representation of the Poisson bracket algebra.

- Consider a subset of wave functions obeying

$$\frac{\partial \Psi}{\partial p} = 0$$

  In this case, $\mathcal{P}(x) = x$, $\mathcal{P}(p) = -i\frac{\partial}{\partial x}$, which still obey $[\mathcal{P}(x), \mathcal{P}(p)] = i$.

  We have a representation on a subspace $\Longrightarrow$ previous representation is reducible.

QUANTIZATION (*cont'd.*)

- Obtain irreducibility by subsidiary conditions on Ψ which restrict its dependence to half the number of variables (Choice of polarization).
- Choose n vector fields $P_i = P_i^\mu (\partial / \partial q^\mu)$, obeying

$$\Omega_{\mu\nu} \, P_i^\mu \, P_j^\nu = 0$$

  and impose

$$P_i^\mu \mathcal{D}_\mu \Psi = 0$$

  The vectors P_i define the polarization. The restricted wave functions are the true wave functions of the theory.

- Integrability conditions for this:

$$[P_i^\mu \mathcal{D}_\mu, P_j^\nu \mathcal{D}_\nu] \, \Psi = 0$$

- This is obtained if

$$[P_i^\mu \frac{\partial}{\partial q^\mu}, P_j^\nu \frac{\partial}{\partial q^\nu}] = C_{ij}^k \, P_k^\alpha \frac{\partial}{\partial q^\alpha}, \qquad \Omega_{\mu\nu} \, P_i^\mu \, P_j^\nu = 0$$

QUANTIZATION (*cont'd.*)

- The true wave functions do not depend on half the number of phase space coordinates, so one cannot integrate using $d\sigma(M)$
- What should be the inner product on the true wave functions?
- Generally difficult, no natural volume measure on restricted subspace of phase space.
- One case where this is possible: M is a Kähler space, Ω is proportional to the Kähler form.
- For a Kähler space,

$$\begin{aligned} \Omega &= \Omega_{a\bar{a}}\, dx^a \wedge d\bar{x}^{\bar{a}} = \frac{i}{2} \partial_a \partial_{\bar{a}} K \;\; dx^a \wedge d\bar{x}^{\bar{a}} = d\mathcal{A} \\ \mathcal{A}_a &= -\frac{i}{2} \partial_a K, \qquad \mathcal{A}_{\bar{a}} \;=\; \frac{i}{2} \partial_{\bar{a}} K \\ \text{Metric } g_{a\bar{a}} &= \partial_a \partial_{\bar{a}} K \end{aligned}$$

QUANTIZATION (*cont'd.*)

- Since $\Omega_{ab} = 0$, choose the (holomorphic or Bargmann) polarization condition

$$\begin{aligned} \mathcal{D}_{\bar{a}}\Psi &= \left(\partial_{\bar{a}} + \tfrac{1}{2}\partial_{\bar{a}}K\right)\Psi = 0 \\ \Psi &= \exp(-\tfrac{1}{2}K)\,F \end{aligned}$$

  F is holomorphic, with $\partial_{\bar{a}}F = 0$.

- The inner product is

$$\langle 1|2\rangle = \int d\sigma(M)\, e^{-K}\, F_1^* F_2$$

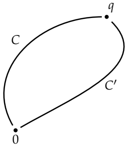
- Operator = Pre-quantum operator subject to polarization if it preserves polarization; otherwise construct matrix element directly.

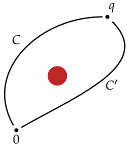
TOPOLOGICAL FEATURES: $\mathcal{H}^1(M,\mathbb{R})$

- Consider $\mathcal{A}$ and $\mathcal{A}+A$ which lead to same Ω,

$$d\mathcal{A}=\Omega, \quad d(\mathcal{A}+A)=\Omega \qquad \Longrightarrow \; dA=0$$

- $A=d\Lambda \Longrightarrow$ remove it by canonical (unitary) transformation, $\Psi \Longrightarrow e^{i\Lambda}\Psi$.
- We can have $dA=0$ with $A\neq d\Lambda$; this means $\mathcal{H}^1(M,\mathbb{R})\neq 0$.
- We can try $\Psi=\exp\left(i\int_0^q A\right)\Phi$.





- The path-dependence of the phase factor:
  - $\int_C A-\int_{C'} A=\oint A=\int_S dA=0$
  - If the path $C-C'$ is noncontractible with no surface S whose boundary is $C-C'$, then $\oint A$ can be nonzero.

TOPOLOGICAL FEATURES: $\mathcal{H}^1(M, \mathbb{R})$

- Using $\Psi = \exp\left(i \int_0^q A\right) \Phi$ eliminates A but Φ need not be single-valued.
- Let $A = \theta\alpha$ where θ is a constant and $\int \alpha = 1$ for a single traversal of the basic noncontractible path corresponding to $C - C'$ (once around the red dot).
- Then for n traversals of the path, $\oint A = \theta\, n$.
- We can eliminate A and use Φ; but Φ is not single-valued and changes by $\exp(i\theta n)$ going around the noncontractible path n times.
- We have an extra constant θ required to define the quantum theory.
- Examples:
  - Fractional statistics in two spatial dimensions
  - Theta vacua in quantum chromodynamics

TOPOLOGICAL FEATURES: $\mathcal{H}^2(M, \mathbb{R})$

- This occurs when we have closed 2-forms which are not exact; i.e., $d\Omega = 0$, but $\Omega \neq d\mathcal{A}$ for any globally defined $\mathcal{A}$.
- Correspondingly, there are two-surfaces which are closed but are not boundaries of any 3-volumes
- If $\Omega = d\mathcal{A}$, with $\mathcal{A}$ well-defined globally, for a closed surface Σ,

$$\int_\Sigma \Omega = \int_{\partial\Sigma} \mathcal{A} = 0$$

- If $\Omega \neq d\mathcal{A}$, the integral of Ω over a closed noncontractible 2-surface can be nonzero.

$$I(\Sigma) = \int_\Sigma \Omega$$

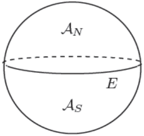
$$I(\Sigma) - I(\Sigma') = \int_{\Sigma - \Sigma'} \Omega = \int_V d\Omega = 0$$

- The integral of Ω over any closed two-surface is a topological invariant, invariant under small deformations of the surface.
- If Σ is contractible, deform Σ to zero $\Longrightarrow \int_\Sigma \Omega = 0$.
- Otherwise, $I(\Sigma)$ can be nonzero.

TOPOLOGICAL FEATURES: $\mathcal{H}^2(M, \mathbb{R})$

- Example of Σ as a two-sphere:
  - Cover the surface with two patches, a northern hemisphere and a southern hemisphere, with $\Omega = d\mathcal{A}_N$ and $\Omega = d\mathcal{A}_S$ on corresponding patches
  - On the overlap region, the equator E,

$$\begin{aligned} \mathcal{A}_N &= \mathcal{A}_S + d\Lambda \\ \Psi_N &= \exp(i\Lambda)\, \Psi_S \end{aligned}$$



$$\Delta\Lambda = \oint_E d\Lambda \; = \; \int_E \mathcal{A}_N - \mathcal{A}_S \; = \; \int_{\partial N} \mathcal{A}_N \; + \; \int_{\partial S} \mathcal{A}_S \; = \; \int_N \Omega \; + \; \int_S \Omega \; = \; \int_\Sigma \; \Omega$$

- Λ is not single-valued on the equator; but Ψ must be. Thus $\exp(i\Delta\Lambda) = 1$, or

$$\int_\Sigma \Omega = 2\pi n, \qquad (\textit{Dirac};\ \textit{Generalized Bohr-Sommerfeld condition})$$

- Examples of this are:
  - Charged particle in a magnetic monopole background
  - Chern-Simons and WZW theories

SUMMARIZING QUANTIZATION

We will consider quantization with the holomorphic polarization.

- A phase space which is also Kähler; the symplectic two-form must be a multiple of the Kähler form.
- The polarization condition is chosen as $\mathcal{D}_{\bar{a}} \Psi \ = \ 0$.
- The inner product of the prequantum Hilbert space = Square integrability on the phase space ⇒ Inner product on the true Hilbert space in the holomorphic polarization.
- $f(q)$ which preserves the polarization ⇒ Prequantum operator $\mathcal{P}(f)$ restricted to the true (polarized) wave functions.
- For observables which do not preserve the polarization, one has to construct infinitesimal unitary transformations whose classical limits are the required canonical transformations.
- If the phase space M has noncontractible two-surfaces, then the integral of Ω over any of these surfaces must be quantized in units of 2π.
- If $\mathcal{H}^1(M, \mathbb{R})$ is not zero, then there are inequivalent $\mathcal{A}$'s for the same Ω and we need extra angular parameters to specify the quantum theory completely.

Quantizing the Two-sphere

- Take the phase space as the two-sphere $S^2 \sim \mathbb{CP}^1 \sim SU(2)/U(1)$.
- This is a Kähler manifold; basic parameters are:

Coordinates	$z = x + iy, \qquad \bar{z} = x - iy$
Kähler two-form	$\omega = i\, dz \wedge d\bar{z}/(1 + z\bar{z})^2$
Metric	$ds^2 = dz\, d\bar{z}/(1 + z\bar{z})^2$
Riemannian curvature	$R_{1\,2} = 4\, dx \wedge dy/(1 + z\bar{z})^2$
Euler number	$\chi = \int (R_{12}/2\pi) = 2$

- S^2 has nontrivial $\mathcal{H}^2(S^2, \mathbb{R})$ given by ω.
- The symplectic two-form is taken as

$$\Omega = n\, \omega = i\, n\, \frac{dz \wedge d\bar{z}}{(1 + z\bar{z})^2}$$

where n is an integer, in agreement with Dirac-Bohr-Sommerfeld condition.

QUANTIZING THE TWO-SPHERE (*cont'd.*)

- The symplectic potential is

$$\begin{aligned} \mathcal{A} &= \frac{in}{2}\left[\frac{z\, d\bar{z} - \bar{z}\, dz}{(1+z\bar{z})}\right] = \frac{i}{2}\partial_{\bar{z}} K\, d\bar{z} - \frac{i}{2}\partial_z K\, dz \\ K &= n \log(1+z\bar{z}) \end{aligned}$$

- Choose the polarization condition as

$$(\partial_{\bar{z}} - i\mathcal{A}_{\bar{z}})\,\Psi = \left[\partial_{\bar{z}} + \frac{n}{2}\frac{z}{1+z\bar{z}}\right]\Psi = 0$$

- This has the solution

$$\Psi = \exp\left(-\frac{n}{2}\log(1+z\bar{z})\right) f(z)$$

  with the inner product

$$\langle 1|2\rangle = i(n+1)\int \frac{dz \wedge d\bar{z}}{2\pi(1+z\bar{z})^{n+2}}\; {f_1}^* f_2$$

- Normalizable states correspond to linear combinations of $f(z) = 1,\ z,\ z^2, \cdots, z^n$; dimension of Hilbert space $= n+1$. (Inner product normalized so that $\mathrm{Tr}(\mathbf{1}) = n+1$.)

QUANTIZING THE TWO-SPHERE (*cont'd.*)

- There are three independent vector fields on S^2 which preserve the metric and ω (Hamiltonian vector fields).

Vector field	Function on phase space
$\xi_+ = i\left(\frac{\partial}{\partial \bar{z}} + z^2 \frac{\partial}{\partial z}\right)$	$J_+ = -n\,\frac{z}{1+z\bar{z}}$
$\xi_- = i\left(\frac{\partial}{\partial z} + \bar{z}^2 \frac{\partial}{\partial \bar{z}}\right)$	$J_- = -n\,\frac{\bar{z}}{1+z\bar{z}}$
$\xi_3 = i\left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\right)$	$J_3 = -\frac{n}{2}\left(\frac{1-z\bar{z}}{1+z\bar{z}}\right)$

- Check one case:

$$\begin{aligned} i_{\xi_+}\Omega &= i(\partial_{\bar{z}} + z^2\partial_z) \lrcorner\; in\frac{dz\wedge d\bar{z}}{(1+z\bar{z})^2} \\ &= -n\left[-\frac{dz}{(1+z\bar{z})^2} + \frac{z^2 d\bar{z}}{(1+z\bar{z})^2}\right] \\ &= -d\left[-\frac{nz}{(1+z\bar{z})}\right] \end{aligned}$$

QUANTIZING THE TWO-SPHERE (*cont'd.*)

- The pre-quantum operators are

$$
\begin{aligned}
\mathcal{P}(J_+) &= \left(z^2 \partial_z - \frac{n\,z}{2} \frac{2 + z\bar{z}}{1 + z\bar{z}} \right) \; - i\xi_+^{\bar{z}} \mathcal{D}_{\bar{z}} \\
\mathcal{P}(J_-) &= \left(-\partial_z - \frac{n}{2} \frac{\bar{z}}{1 + z\bar{z}} \right) - i\xi_-^{\bar{z}} \mathcal{D}_{\bar{z}} \\
\mathcal{P}(J_3) &= \left(z\partial_z - \frac{n}{2} \frac{1}{1 + z\bar{z}} \right) - i\xi_3^{\bar{z}} \mathcal{D}_{\bar{z}}
\end{aligned}
$$

- On the polarized wave functions, $\mathcal{D}_{\bar{z}} \Psi = 0$, giving the quantum operators acting on $f(z)$,

$$
\begin{aligned}
\hat{J}_+ &= z^2 \partial_z - n\,z \\
\hat{J}_- &= -\partial_z \\
\hat{J}_3 &= z\partial_z - \tfrac{1}{2}\, n
\end{aligned}
$$

- These obey $SU(2)$ algebra.
- The full Hilbert space corresponds to one UIR of $SU(2)$ with $j = n/2$.

QUANTIZING THE TWO-SPHERE (*cont'd.*)

- The form of the action is

$$\begin{aligned}\mathcal{S} &= \int dt\, \mathcal{A}_\mu \frac{dq^\mu}{dt} = i\frac{n}{2}\int dt\, \frac{z\dot{\bar{z}} - \bar{z}\dot{z}}{1+z\bar{z}}\\ &= i\frac{n}{2}\int dt\; \mathrm{Tr}(\sigma_3\, g^{-1}\dot{g})\end{aligned}$$

$g \in SU(2)$; explicitly

$$g = \frac{1}{\sqrt{1+z\bar{z}}}\begin{bmatrix} 1 & z \\ -\bar{z} & 1 \end{bmatrix}\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

QUANTIZING THE G/H SPACE

- More generally, one can take, for $g \in G$,

$$\mathcal{S} = i \sum_a w_a \int dt \, \mathrm{Tr}(t^a \, g^{-1} \dot{g}), \qquad \mathcal{A}(g) = i \sum_a w_a \mathrm{Tr}(t^a g^{-1} dg)$$

Weights of a UIR

Diagonal Generators

- Ω is a two-form on G/H, H = maximal subgroup of G commuting with $\sum_a w_a t^a$.
- Consistent quantization ($\int \Omega = 2\pi n$) requires that $\{w_s\}$ must be the highest weights for a unitary irreducible representation (UIR) of G.
- Upon quantization, this action gives exactly one unitary irreducible representation (UIR) of G, namely the one corresponding to the highest weight state $(w_1, w_2, \cdots)$.

THE CHERN-SIMONS THEORY IN 2+1 DIMENSIONS

- The action is given by

$$\begin{aligned}\mathcal{S} &= -\frac{k}{4\pi}\int_{\Sigma\times[t_i,t_f]} \mathrm{Tr}\left[A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right]\\ &= -\frac{k}{4\pi}\int_{\Sigma\times[t_i,t_f]} d^3x\; \epsilon^{\mu\nu\alpha}\,\mathrm{Tr}\left[A_\mu\partial_\nu A_\alpha + \frac{2}{3}A_\mu A_\nu A_\alpha\right]\end{aligned}$$

  Σ is usually taken as a Riemann surface.

- Choose $A_0 = 0$ as a gauge condition; then

$$\mathcal{S} = -\frac{ik}{\pi}\int dt d\mu_\Sigma\, \mathrm{Tr}(A_{\bar{z}}\,\partial_0 A_z) \qquad \Longrightarrow \qquad \mathcal{A} \;=\; -\frac{ik}{\pi}\int_\Sigma \mathrm{Tr}\big(A_{\bar{z}}\delta A_z\big) \;+\; \delta\rho[A]$$

- The symplectic two-form is

$$\Omega = -\frac{ik}{\pi}\int_\Sigma d\mu_\Sigma\,\mathrm{Tr}\big(\delta A_{\bar{z}}\delta A_z\big) = \frac{ik}{2\pi}\int_\Sigma d\mu_\Sigma\;\delta A^a_{\bar{z}}\delta A^a_z$$

- The space of 2-d gauge potentials is Kähler with the Kähler potential

$$K \;=\; \frac{k}{2\pi}\int_\Sigma A^a_{\bar{z}}A^a_z$$

THE CHERN-SIMONS THEORY IN 2+1 DIMENSIONS (*cont'd.*)

- (Time-independent) gauge transformations act on the potentials as

$$A^g = gAg^{-1} - dgg^{-1} \approx A - D\theta \qquad \text{infinitesimally}$$

- The infinitesimal transformations are generated by the vector field

$$\xi = -\int_\Sigma \left((D_z\theta)^a \frac{\delta}{\delta A^a_z} + (D_{\bar z}\theta)^a \frac{\delta}{\delta A^a_{\bar z}}\right)$$

  Acting on Ω we get

$$\begin{aligned} i_\xi\Omega &= -\int \left((D_z\theta)^a \frac{\delta}{\delta A^a_z} + (D_{\bar z}\theta)^a \frac{\delta}{\delta A^a_{\bar z}}\right) \rfloor \frac{ik}{2\pi}\int_\Sigma d\mu_\Sigma\, \delta A^a_{\bar z}\delta A^a_z \\ &= -\frac{ik}{2\pi}\int \left[((\bar D\theta)^a \delta A^a_z - (D\theta)^a \delta A^a_{\bar z}\right] = \frac{ik}{2\pi}\int \theta^a(\bar D\delta A_z - D\delta A_{\bar z})^a \\ &= \frac{ik}{2\pi}\int \theta^a \delta F^a_{\bar z z} = -\delta\left[\int \theta^a \frac{ik}{2\pi} F^a_{z\bar z}\right] \end{aligned}$$

- The generator of gauge transformations is

$$G^a = \frac{ik}{2\pi} F^a_{z\bar z}$$

  This has to vanish on wave functions, $G^a\Psi = 0$.

THE CHERN-SIMONS THEORY IN 2+1 DIMENSIONS (*cont'd.*)

- The prequantum wave functions have the inner product

$$(1|2) \;=\; \int d\mu(A_z, A_{\bar z})\; \Psi_1^*[A_z, A_{\bar z}]\; \Psi_2[A_z, A_{\bar z}]$$

- The symplectic potential is

$$\mathcal{A} \;=\; -\frac{ik}{2\pi}\int_\Sigma \mathrm{Tr}\big(A_{\bar z}\delta A_z \;-\; A_z\delta A_{\bar z}\big) \;=\; \frac{ik}{4\pi}\int_\Sigma \big(A^a_{\bar z}\delta A^a_z - A^a_z\delta A^a_{\bar z}\big)$$

- The covariant derivatives with $\mathcal{A}$ as the potential are

$$\nabla \;=\; \frac{\delta}{\delta A^a_z} + \frac{k}{4\pi}A^a_{\bar z}, \qquad \overline{\nabla} \;=\; \frac{\delta}{\delta A^a_{\bar z}} - \frac{k}{4\pi}A^a_z$$

- The Bargmann polarization condition is $\nabla\Psi = 0$, with the solution

$$\Psi \;=\; \exp\left(-\frac{k}{4\pi}\int A^a_{\bar z}A^a_z\right)\,\psi[A^a_{\bar z}] \;=\; e^{-\frac{1}{2}K}\,\psi[A^a_{\bar z}]$$

ψ's are antiholomorphic, depend only on $A_{\bar z}$'s.

THE CHERN-SIMONS THEORY IN 2+1 DIMENSIONS (*cont'd.*)

- The inner product is now

$$\langle 1|2\rangle \;=\; \int [dA^a_{\bar z} dA^a_z]\, e^{-K(A^a_{\bar z}, A^a_z)}\ \psi_1^*\ \psi_2$$

- On the (anti)holomorphic part ψ of the wave functionals

$$A^a_z\ \psi[A^a_{\bar z}] = \frac{2\pi}{k}\frac{\delta}{\delta A^a_{\bar z}}\ \psi[A^a_{\bar z}]$$

- The condition of $G^a \Psi = 0$ thus becomes

$$\left(D_{\bar z}\,\frac{\delta}{\delta A^a_{\bar z}} \;-\; \frac{k}{2\pi}\partial_z A^a_{\bar z} \right)\ \psi[A^a_{\bar z}]\ = 0.$$

- Before solving this, we consider quantization of k.

THE CHERN-SIMONS THEORY IN 2+1 DIMENSIONS (*cont'd.*)

- Construct a noncontracible two-surface in the configuration space. Start with the loop of gauge transformations

$$C = g(x,\lambda), \qquad 0 \leq \lambda \leq 1, \qquad\quad g(x,0) = g(x,1) \;=\; 1$$

- We then define

$$A(x,\lambda,\sigma) \;=\; (gAg^{-1} - dgg^{-1})\,\sigma \;+\; (1-\sigma)A$$

  where $0 \leq \sigma \leq 1$.

- This goes to A at $\lambda = 0,\ 1$ and at $\sigma = 0$. Further, $A \longrightarrow A^g$ at $\sigma = 1$. Thus this is a closed two-surface in $\mathfrak{C} = \mathfrak{F}/\mathfrak{G}_*$.

THE CHERN-SIMONS THEORY IN 2+1 DIMENSIONS (*cont'd.*)

- For simplicity, take the starting point as $A = 0$ to get

$$A(x, \lambda, \sigma) \;=\; -\; dg\, g^{-1}\, \sigma$$

$$\delta A(x, \lambda, \sigma) = g\, d(g^{-1}\delta g)\, g^{-1}\sigma \;+\; dg\, g^{-1} d\sigma$$

- The integral of Ω over this surface is

$$\begin{aligned}
\int \Omega \;&=\; \frac{k}{4\pi} \int \mathrm{Tr}(\delta A \wedge \delta A) \\
&=\; \frac{k}{4\pi} 2 \int \mathrm{Tr}\left[d(g^{-1}\delta g) g^{-1} dg\right] \int \sigma d\sigma \\
&=\; -2\pi\, k\, Q[g] \\
Q[g] \;&=\; \frac{1}{24\pi^2} \int \mathrm{Tr}(dgg^{-1})^3
\end{aligned}$$

$Q[g] =$ Winding number of the map $g : S^3 \rightarrow G \;\in \mathbb{Z}$

Dirac condition $\Longrightarrow$ k must be an integer.

THE WESS-ZUMINO-WITTEN THEORY

- This is defined by an action functional in 2 Euclidean (or $1+1$) dimensions,

$$\mathcal{S}_{\rm WZW} = \frac{1}{8\pi}\int_{\mathcal{M}^2} d^2x\sqrt{g}\, g^{ab}\, {\rm Tr}(\partial_a M\, \partial_b M^{-1}) \; + \; \Gamma[M]$$

$$\Gamma[M] = \frac{i}{12\pi}\int_{\mathcal{M}^3} d^3x\, \epsilon^{\mu\nu\alpha}\, {\rm Tr}(M^{-1}\partial_\mu M\, M^{-1}\partial_\nu M\, M^{-1}\partial_\alpha M) = \frac{i}{12\pi}\int_{\mathcal{M}^3} {\rm Tr}(M^{-1}dM)^3$$

  $M(x) \in GL(N, \mathbb{C})$ (or suitable subgroups)

- $\Gamma[M]$ = Wess-Zumino term, defined by integration over $\mathcal{M}^3$ with $\partial\mathcal{M}^3 = \mathcal{M}^2$.
- Many $\mathcal{M}^3$'s with the same boundary $\mathcal{M}^2$ possible $\equiv$ Different ways to extend $M(x)$ to $\mathcal{M}^3$.
- If M and M' are two different extensions of the same field, then $M' = MN$, with $N = 1$ on $\mathcal{M}^2$,

$$\Gamma[MN] = \Gamma[M] + \Gamma[N] - \frac{i}{4\pi}\int_{\mathcal{M}^2} d^2x\, \epsilon^{ab}{\rm Tr}\,\underbrace{(M^{-1}\partial_a M\, \partial_b N N^{-1})}_{=0}$$

$N = 1$ on $\partial\mathcal{M}^3 \Longrightarrow N$ is (equivalent to) a map $N:\; S^3 \to G$, classified by $\Pi_3(G)$ (or $Q[N]$).

THE WESS-ZUMINO-WITTEN THEORY (*cont'd.*)

- Independence of the extension follows from:

  1. $\Gamma[N] \;=\; 0$ for $N \approx 1$ (to linear order in $\partial N N^{-1}$).

     By successive transformations, $\Gamma[M]$ is independent of the extension to $\mathcal{M}^3$ for all N connected to identity.

  2. If N is homotopically nontrivial, $\Gamma[N] = 2\pi i\, Q[N]$

     ($\exp(-k\,\Gamma[M])$ is independent of the extension, if $k \in \mathbb{Z}$. So $\mathcal{S} = k\,\mathcal{S}_{WZW}$ can be used as the action for a theory, the WZW theory with level number k.)

- In complex coordinates

$$\mathcal{S}_{WZW} = \frac{1}{2\pi}\int_{\mathcal{M}^2} \mathrm{Tr}(\partial_z M\,\partial_{\bar{z}} M^{-1}) \;+ \Gamma[M]$$

$$\mathcal{S}_{WZW}[M\,h] = \mathcal{S}_{WZW}[M] + \mathcal{S}_{WZW}[h] - \frac{1}{\pi}\int_{\mathcal{M}^2} \mathrm{Tr}(M^{-1}\partial_{\bar{z}} M\,\partial_z h\, h^{-1})$$

(Polyakov-Wiegmann identity)

- Chiral splitting: Antiholomorphic derivative of M, holomorphic derivative of h

THE WESS-ZUMINO-WITTEN THEORY (*cont'd.*)

- Another important property $M \longrightarrow M + \delta M = (1+\theta)M, \;\; \theta = \delta M\, M^{-1}$ infinitesimal.

$$\begin{aligned}
\delta \mathcal{S}_{WZW} &= -\frac{1}{\pi}\int \mathrm{Tr}\left(\partial_{\bar z}(\delta M M^{-1})\partial_z M M^{-1}\right)\\
&= -\frac{1}{\pi}\int \mathrm{Tr}(\delta M M^{-1}\partial_{\bar z} A_z)\\
&= -\frac{1}{\pi}\int \mathrm{Tr}(\delta M M^{-1} D_z \bar{C})\\
&= -\frac{1}{\pi}\int \mathrm{Tr}(\bar{C}\,\delta A_z) = \frac{1}{2\pi}\bar{C}^a \delta A_z^a
\end{aligned}$$

$A_z = -\partial_z M M^{-1}, \quad \bar{C} = -\partial_{\bar z} M\, M^{-1}$

$$D_z \bar{C} = \partial_z \bar{C} + [A_z, \bar{C}]$$

- A_z and $\bar{C}$ obey the equation

$$\partial_{\bar z} A_z - \partial_z \bar{C} \; + \; [\bar{C}, A_z] = 0, \qquad D_z\left[\frac{\delta \mathcal{S}_{WZW}}{\delta A_z}\right] = \frac{1}{2\pi}\partial_{\bar z} A_z$$

This will be useful for evaluating Dirac determinants.

THE WESS-ZUMINO-WITTEN THEORY (*cont'd.*)

- If we use $M^\dagger$, we get C rather than $\bar{C}$.

$$D_z \frac{\delta \mathcal{S}_{WZW}}{\delta A^a_{\bar{z}}} = \frac{1}{2\pi} \partial_z A_{\bar{z}}$$

- Comparing with wave function for CS theory,

$$\psi[\bar{A}] = \exp\left[k\, \mathcal{S}_{WZW}(M^\dagger)\right]$$

provided we can parametrize a general 2-dimensional gauge field as $A_z = -\partial_z M\, M^{-1}$.

THE WESS-ZUMINO-WITTEN THEORY (*cont'd.*)

- A parametrization for gauge potentials

$$A_z = -\partial_z M\, M^{-1} \qquad\qquad A_{\bar{z}} = M^{\dagger\,-1} \partial_{\bar{z}} M^\dagger$$

M is a complex matrix. ($\det M = 1$ if gauge group is $SU(N)$.)

- For $U(1)$, use elementary result $A_i = \partial_i \theta + \epsilon_{ij} \partial_j \phi$. $\quad \Longrightarrow M = \exp(\phi + i\,\theta)$.
- One can invert ∂_z via

$$\left(\frac{1}{\partial_z}\right)_{xx'} = \frac{1}{\pi(\bar{z} - \bar{z}')}$$

- Write $\partial_z M = -A_z M$,

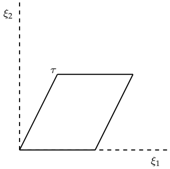
$$\begin{aligned} M(x) &= 1 - \int_{x'} \left(\frac{1}{\partial_z}\right)_{xx'} A_z(x') M(x') \\ &= 1 - \int (\partial_z)^{-1} A_z + \int (\partial_z)^{-1} A_z (\partial_z)^{-1} A_z + \cdots \end{aligned}$$

- The real advantage is that gauge transformations are homogeneous in terms of M,

$$A \to A^g = g A g^{-1} - dg\, g^{-1} \Longrightarrow M^g = gM$$

THE WESS-ZUMINO-WITTEN THEORY (*cont'd.*)

- Comment: Space not simply connected $\Longrightarrow \exists$ zero modes for $\partial_{\bar{z}}$ $\Longrightarrow \exists$ flat potentials a, not gauge equivalent to zero.
- Example: Torus $S^1 \times S^1$. Real coordinates $\xi_1,\ \xi_2$, $0 \leq \xi_i \leq 1$, with $\xi_1 = 0 \ \sim\ \xi_1 = 1$, same for ξ_2.



$z = \xi_1 + \tau \xi_2, \quad \tau =$ modular parameter

- For the torus,the generalized parametrization is

$$A_{\bar{z}} = M \left[\frac{i\pi\, a}{\text{Im}\ \tau} \right] M^{-1} - \partial_{\bar{z}} M\, M^{-1}$$

- Ambiguity: M and $M V(\bar{z}) \Longrightarrow$ same $A_{\bar{z}}$. (Must ensure this does not affect physical results)

θ-VACUA IN 3+1 DIMENSIONS

Analyze topology and geometry of the space of gauge fields in a Hamiltonian description

- Choose $A_0 = 0$ gauge; we are then left with the spatial components $A_i(x)$ which are Lie-algebra-valued vector fields on space.
- A gauge transformation acts on A_i as $A_i \to A_i^g = g^{-1} A_i g + g^{-1} \partial_i g, \;\; g \in G$.
- Define

$$\begin{aligned}
\tilde{\mathfrak{F}} \quad &\equiv \quad \{\text{Set of all gauge potentials } A_i\} \\
&\equiv \quad \{\text{Set of all Lie} - \text{algebra} - \text{valued vector fields on space } \mathbb{R}^d\} \\
\mathfrak{G} \quad &\equiv \quad \{\text{Set of all } g(\vec{x}) : \mathbb{R}^d \to G, \; \text{such that } g(\vec{x}) \longrightarrow \; \text{constant} \in G \text{ as } |\vec{x}| \longrightarrow \infty\} \\
\mathfrak{G}_* \quad &\equiv \quad \{\text{Set of all } g(\vec{x}) : \mathbb{R}^d \to G, \; \text{such that } g(\vec{x}) \longrightarrow 1 \text{ as } |\vec{x}| \longrightarrow \infty\}
\end{aligned}$$

- Evidently $\mathfrak{G}/\mathfrak{G}_* = G$. This acts as a Noether symmetry classifying charged states in the theory.
- $\mathfrak{G}_*$ is the true gauge symmetry, with A_i and A_i^g physically equivalent for $g(x) \in \mathfrak{G}_*$.

θ-VACUA IN 3+1 DIMENSIONS (*cont'd.*)

- The physical configuration space is $\mathfrak{C} = \tilde{\mathfrak{F}}/\mathfrak{G}_*$
- Consider $2+1$ dimensions

$$\Pi_2(\mathfrak{C}) = \Pi_1(\mathfrak{G}_*) = \Pi_3(G) = \begin{cases} \mathbb{Z} & \text{All compact } G \neq SO(4) \\ \mathbb{Z}\times\mathbb{Z} & G = SO(4) \end{cases}$$

- How does this arise?
  - An element of $\mathfrak{G}_*$ is $g(\vec{x})$ with $g \to 1$ at spatial infinity $\Rightarrow$ $\Pi_0(\mathfrak{G}_*) = \Pi_2(G) = 0$.
  - For connectivity, examine closed paths starting and ending at $g(\vec{x}) = 1$. Such a path is given by $g(\vec{x}, \lambda)$; $0 \leq \lambda \leq 1$ parametrizes path, with $g(\vec{x}, 0) = g(\vec{x}, 1) = 1$.
  - $g(\vec{x}, \lambda) : \mathbb{R}^3 \to G$ with $g \to 1$ at the 'boundary'. This is equivalent to a map from S^3 to G, classified by $\Pi_3(G)$.
- There are noncontractible two-surfaces in $\mathcal{C}$ and hence in the phase space.

Gauge theories in $2+1$ dimensions have $\mathcal{H}^2(M, \mathbb{R}) \neq 0$; they can show Dirac quantization conditions (depending on choice of Ω)

θ-VACUA IN 3+1 DIMENSIONS (*cont'd.*)

- Consider $3+1$ dimensions

$$\Pi_1(\mathfrak{C}) = \Pi_0(\mathfrak{G}_*) = \Pi_3(G) = \begin{cases} \mathbb{Z} & \text{All compact simple } G \neq SO(4) \\ \mathbb{Z}\times\mathbb{Z} & G = SO(4) \end{cases}$$

- How does this arise? Similar reasoning as for $2+1$ dimensions
- There are noncontractible paths in $\mathcal{C}$ and hence in phase space.
- The phase space is multiply connected with connectivity given by $\mathbb{Z}$ (or $\mathbb{Z}\times\mathbb{Z}$ for $SO(4)$).

Gauge theories in $3+1$ dimensions have $\mathcal{H}^1(M,\mathbb{R}) \neq 0$; the quantum theory will require additional vacuum angles (θ-vacua) to characterize it.

θ-VACUA IN 3+1 DIMENSIONS (*cont'd.*)

- Start with the Yang-Mills action and choose $A_0 = 0$,

$$\mathcal{S} = \frac{1}{4} \int d^4x \, F^a_{\mu\nu} F^{a\mu\nu} \;=\; \frac{1}{2} \int d^4x \, \underbrace{(\partial_0 A^a_i)}_{E^a_i}(\partial_0 A^a_i) \;+\; \cdots$$

- The symplectic potential is $\mathcal{A} = \int d^3x \, E^a_i \, \delta A^a_i$ and

$$\Omega = \int d^3x \, \delta E^a_i \, \delta A^a_i \;=\; -2 \int d^3x \, \mathrm{Tr}\,(\delta E_i \, \delta A_i)$$

  The condition of gauge invariance (under $g \approx 1 + \varphi$) is the Gauss law given by

$$G(\varphi)\Psi = \int d^3x \, \varphi^a (D_i E_i)^a \, \Psi = 0$$

- An element of $\mathfrak{G}_*$ is a map $g(x) : \mathbb{R}^3 \to G$ with the condition $g \to 1$ at spatial infinity.

  These are equivalent to maps $S^3 \to G$ and are characterized by the winding number $Q[g]$.

$$\mathfrak{G}_* = \sum_{Q=-\infty}^{+\infty} \oplus \mathfrak{G}_{*Q}$$

  This leads to $\Pi_1(\mathfrak{C}) = \mathbb{Z}$.

θ-VACUA IN 3+1 DIMENSIONS (*cont'd.*)

- Construct a one-form on $\mathfrak{C}$ which is closed but not exact.

$$K[A] = -\frac{1}{4\pi^2}\int \mathrm{Tr}(F \wedge \delta A) = \frac{1}{16\pi^2}\int d^3x\, \epsilon^{ijk}\, F^a_{jk}\, \delta A^a_i$$

  - Closure: $K[A] = \delta(\mathcal{S}_{CS}/2\pi)$, so using $\delta^2 = 0$, $\delta K = 0$
  - But K is not exact, even though $K = \delta(\mathcal{S}_{CS}/2\pi)$, because $\mathcal{S}_{CS}$ is not gauge-invariant. It is not a function on $\mathcal{C}$.
  - $K[A]$ is the generating element of $\mathcal{H}^1(\mathfrak{C}, \mathbb{R})$.

- An example of the noncontractible loop:

$$A_i(x,\tau) = (g\, A_i\, g^{-1} - \partial_i g\, g^{-1})\tau + A_i(x)(1-\tau), \qquad 0 \leq \tau \leq 1$$

This is an open path in $\mathfrak{F}$; the end-points are gauge transforms of each other, so it is closed in $\mathfrak{C}$. If the path is contractible, it is deformable to

$$A_i(x,\tau) = A(x)^{g(x,\tau)}, \qquad g(x,0) = 1, \quad g(x,1) = g(x)$$

$g(x,\tau)$ makes $g(x)$ homotopic to $g = 1$. This is not possible if $Q[g] \neq 0$.

θ-VACUA IN 3+1 DIMENSIONS (*cont'd.*)

- Integrate K along such a curve,

$$\begin{aligned} \oint K[A] &= \frac{1}{2\pi}\Big(\mathcal{S}_{CS}[A^g] - \mathcal{S}_{CS}[A]\Big) \\ &= -\frac{1}{8\pi^2}\int \mathrm{Tr}(F \wedge F) \qquad \text{(Instanton number)} \\ &= -\frac{1}{32\pi^2}\int d^4x\ \mathrm{Tr}(F_{\mu\nu}F_{\alpha\beta})\epsilon^{\mu\nu\alpha\beta} \end{aligned}$$

- Since $\delta K = 0$, we get the same Ω for $\mathcal{A}$ and $\mathcal{A} + \theta K$.

$$\mathcal{A} = \int d^3x\ E_i^a \delta A_i^a \ + \ \theta\ K[A]$$

We need an additional parameter θ to characterize the quantum theory.

- $\oint K$ is an integer, so we can take $0 \leq \theta \leq 2\pi$.
- This is equivalent to using

$$\mathcal{S} = \mathcal{S}_{YM} \ + \ \theta\left[-\frac{1}{8\pi^2}\int \mathrm{Tr}(F \wedge F)\right]$$

FRACTIONAL QUANTUM HALL EFFECT & FRACTIONAL STATISTICS

- For the states with filling fractions $\nu = 1/(2p+1)$ where p is an integer, the N-electron wave function is the Laughlin function

  $$\Psi_{Laughlin} = \mathcal{N} \exp\left(-\tfrac{1}{2}\sum_{i=1}^{N} \bar{z}_i z_i\right) \prod_{1\le i<j\le N} (z_i - z_j)^{2p+1}$$

  where $z = x_1 + i x_2$.
- This leads to an electric current of the form

  $$\langle J_i \rangle = -\nu \frac{e^2}{2\pi} \epsilon_{ij} E_j, \qquad \nu = \frac{1}{2p+1}$$

  This corresponds to the observed Hall conductivity, quantized as the reciprocals of odd integers.
- Among the excited states of the system as hole-like excitations with a wave function of the form

  $$\Psi_{hole} = \prod_{i=1}^{N} (z_i - w)\Psi_{Laughlin} = \prod_{i=1}^{N} (z_i - w)\, \mathcal{N} \exp\left(-\tfrac{1}{2}\sum_{i=1}^{N} \bar{z}_i z_i\right) \prod_{1\le i<j\le N} (z_i - z_j)^{2p+1}$$

  where w is the position of the hole.

FRACTIONAL QUANTUM HALL EFFECT & FRACTIONAL STATISTICS (*cont'd.*)

- We can consider statistics of holes using an effective action of the form

$$S = \int d^3x \left[\frac{k}{4\pi} \epsilon^{\mu\nu\alpha} a_\mu \partial_\nu a_\alpha + a_\mu \left(j^\mu - \frac{e}{2\pi} \epsilon^{\mu\nu\alpha} \partial_\nu A_\alpha \right) \right]$$

- The electromagnetic current is

$$J^\alpha = -\frac{e}{2\pi} \epsilon^{\alpha\mu\nu} \partial_\mu a_\nu$$

  where J^μ denotes electromagnetic current.

- The equation of motion for the auxiliary field a_μ is

$$\frac{k}{2\pi} \epsilon^{\mu\nu\alpha} \partial_\nu a_\alpha + j^\mu - \frac{e}{2\pi} \epsilon^{\mu\nu\alpha} \partial_\nu A_\alpha = 0$$

- We then see that

$$J^\mu = \frac{e}{k} j^\mu - \frac{e^2}{2\pi k} \epsilon^{\mu\nu\alpha} \partial_\nu A^\alpha .$$

  Choosing $k = 2p + 1$ we see that we can reproduce the Hall conductivity correctly in the absence of holes.

- The first term then shows that the charge per hole is e/k.

FRACTIONAL QUANTUM HALL EFFECT & FRACTIONAL STATISTICS (*cont'd.*)

- For a pair of well-separated holes we can take

$$j^\mu = \dot{w}_1^\mu \, \delta^{(2)}(x - w_1) + \dot{w}_2^\mu \, \delta^{(2)}(x - w_2)$$

- Focusing just on the holes, the action becomes

$$S_{hole} = \frac{k}{4\pi} \int d^3x \, \epsilon^{\mu\nu\alpha} a_\mu \partial_\nu a_\alpha + \int dt \, \left(a_\mu(w_1)\dot{w}_1{}^\mu + a_\mu(w_2)\dot{w}_2{}^\mu + \frac{m\dot{w}_1{}^2}{2} + \frac{m\dot{w}_2{}^2}{2} \right)$$

- The time-component of the equation of motion for for a_μ is

$$\partial_z \alpha_{\bar{z}} - \partial_{\bar{z}} a_z = -i \frac{\pi}{k} \left(\delta^{(2)}(x - w_1) + \delta^{(2)}(x - w_2) \right)$$

with the solution

$$a_{\bar{z}} = -\frac{i}{2k} \left(\frac{1}{\bar{z} - \bar{w}_1} + \frac{1}{\bar{z} - \bar{w}_2} \right), \qquad a_z = \frac{i}{2k} \left(\frac{1}{z - w_1} + \frac{1}{z - w_2} \right)$$

The coincident point $w_1 = w_2$ has to be excluded for consistency. We also used

$$\partial_z \frac{1}{\bar{z} - \bar{w}} = \partial_{\bar{z}} \frac{1}{z - w} = \pi \, \delta^{(2)}(x - w)$$

FRACTIONAL QUANTUM HALL EFFECT & FRACTIONAL STATISTICS (*cont'd.*)

- We will also use the $a_0 = 0$ gauge so that the action for the holes simplifies to

$$S = \int dt \left[\frac{m}{2} (\dot{\bar{w}}_1 \dot{w}_1 + \dot{\bar{w}}_2 \dot{w}_2) + a_{w_1} \dot{w}_1 + a_{\bar{w}_1} \dot{\bar{w}}_1 + a_{w_2} \dot{w}_2 + a_{\bar{w}_2} \dot{\bar{w}}_2 \right]$$

  where we have removed the singularities at the poles. Thus

$$\begin{aligned} a_{w_1} &= \frac{i}{2k} \frac{1}{w_1 - w_2}, & a_{\bar{w}_1} &= -\frac{i}{2k} \frac{1}{\bar{w}_1 - \bar{w}_2} \\ a_{w_2} &= \frac{i}{2k} \frac{1}{w_2 - w_1}, & a_{\bar{w}_2} &= -\frac{i}{2k} \frac{1}{\bar{w}_2 - \bar{w}_1} \end{aligned}$$

- The coincident point $w_1 = w_2$ has been excluded, so the closed path of one hole going around the other is *not contractible*. $\Longrightarrow \Pi_1(\text{configuration space}) = \mathbb{Z} \neq 0$.
- With w_2 fixed,

$$d\,a = 0 \quad \text{for} \quad a = a_{w_1} dw_1 + a_{\bar{w}_1} d\bar{w}_1 = d \left[\frac{i}{2k} \log \left(\frac{w_1 - w_2}{\bar{w}_1 - \bar{w}_2} \right) \right]$$

- a is not exact since

$$\oint_C a = -\frac{2\pi}{k} \neq 0, \qquad C \text{ encloses } w_2$$

FRACTIONAL QUANTUM HALL EFFECT & FRACTIONAL STATISTICS (*cont'd.*)

- The Hamiltonian corresponding to the action for holes is

$$H = \frac{1}{2} m \left(\dot{\bar{w}}_1 \dot{w}_1 + \dot{\bar{w}}_2 \dot{w}_2 \right)$$

- From the action we also identify the operators

$$\begin{aligned} m\dot{w}_1 &= -i \frac{\partial}{\partial \bar{w}_1} - a_{\bar{w}_1}, &\qquad m\dot{\bar{w}}_1 &= -i \frac{\partial}{\partial w_1} - a_{w_1} \\ m\dot{w}_2 &= -i \frac{\partial}{\partial \bar{w}_2} - a_{\bar{w}_2}, &\qquad m\dot{\bar{w}}_2 &= -i \frac{\partial}{\partial w_2} - a_{w_2} \end{aligned}$$

- Write the wave function as

$$\Psi(x_1, x_2) = \exp \left[\frac{1}{2k} \log \left(\frac{\bar{w}_1 - \bar{w}_2}{w_1 - w_2} \right) \right] \Phi(x_1, x_2)$$

- The action of H on Φ is

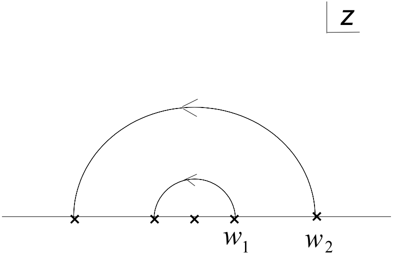
$$H\,\Phi = -\frac{1}{2m} \left(\frac{\partial}{\partial w_1} \frac{\partial}{\partial \bar{w}_1} + \frac{\partial}{\partial w_2} \frac{\partial}{\partial \bar{w}_2} \right) \Phi$$

FRACTIONAL QUANTUM HALL EFFECT & FRACTIONAL STATISTICS (*cont'd.*)

- We can consider the exchange of the two holes as due to a rotation of the two points by π followed by a translation to bring them back to the same points. We take Φ to be symmetric under exchange. As for the phase factor the translation does not change it. The π-rotation leads to

  $$\Psi(x_2, x_1) = e^{-i\pi/k}\, \Psi(x_1, x_2)$$

  With $k = 2p + 1$, we see that the two holes do display fractional statistics.



FLUID DYNAMICS

- Lagrange's approach
  - Newton's equations for N point-particles → coarse graining using a smooth density function → fluid dynamics
- Point particle $\equiv$ a unitary irreducible representation (UIR) of the Poincaré group
- Classical action which upon quantization gives a UIR of a group = A co-adjoint orbit action

> Can we construct fluid dynamics as
>
> Co-adjoint orbit action → coarse graining → fluid dynamics ?

- Advantages:
  - A single formalism where symmetries are foundational
  - Gauge fields → Abelian and nonabelian Magnetohydrodynamics
  - Spin, magnetic moment effects
  - Gravity easily included (Mathisson-Papapetrou equation)
  - Anomalous symmetries (chiral magnetic effect, chiral vorticity effect, etc.)

THE RELATIVISTIC POINT-PARTICLE

- For relativistic point-particles, we must use this action with $G =$ Poincaré group, the group of translations and Lorentz transformations
- We consider Poincaré group = contraction of de Sitter group; this makes some traces easier to define.
- For de Sitter algebra, use standard Dirac γ-matrices with

$$J_{\mu\nu} = \frac{1}{4\,i}[\gamma_\mu, \gamma_\nu], \qquad P_\mu = \frac{\gamma_\mu}{r_0}, \qquad \text{Poincaré} = r_0 \to \infty \text{ limit}$$

- These obey the commutation rules

$$\begin{aligned}
[J_{\mu\nu}, J_{\alpha\beta}] &= i\left(\eta_{\mu\alpha}\, J_{\nu\beta} - \eta_{\mu\beta}\, J_{\nu\alpha} - \eta_{\nu\alpha}\, J_{\mu\beta} + \eta_{\nu\beta}\, J_{\mu\alpha}\right)\\
[J_{\mu\nu}, P_\alpha] &= i\,(\eta_{\mu\alpha}\, P_\nu - \eta_{\nu\alpha}\, P_\mu)\\
[P_\mu, P_\nu] &= i\,\frac{4}{r_0^2}\; J_{\mu\nu}
\end{aligned}$$

- As $r_0 \to \infty$ we get the Poincaré limit.

THE RELATIVISTIC POINT-PARTICLE (*cont'd.*)

- A general element is given by

$$g = \exp\left(i\gamma_\alpha\, x^\alpha / r_0\right)\ \Lambda, \qquad \Lambda = B(p)\, R$$

$$B(p) = \frac{1}{\sqrt{2m(p_0+m)}} \begin{bmatrix} p_0 + m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & p_0 + m \end{bmatrix}$$

Λ is an element of the Lorentz group, R is a pure spatial rotation generated by J_{12}, J_{23}, J_{31}, and $m = \sqrt{p^2}$.

THE RELATIVISTIC POINT-PARTICLE (*cont'd.*)

- The action is given by

$$S = i\, m\, r_0^2 \int d\tau \,\mathrm{Tr}\left(\frac{\gamma_0}{r_0} g^{-1} \frac{dg}{d\tau} \right) + i\, \frac{n}{2}\, \mathrm{Tr}(J_{12}\, g^{-1}\, dg) \;-\; \mathcal{H}$$

  Using $B\gamma_0 B^{-1} = \gamma^\alpha\, p_\alpha / m$ and taking $r_0 \to \infty$, we find, for the Poincaré group,

$$S = -\int d\tau\, p_\mu\, \dot{x}^\mu + i\, \frac{n}{4} \int d\tau\, \mathrm{Tr}(\Sigma_3\, \Lambda^{-1}\, \dot{\Lambda}) \;-\; \mathcal{H}, \qquad \Sigma_3 = \begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix}$$

- $\mathcal{H}$ generates τ-evolution, so we should set it to zero as a constraint on quantum states. This leads to the wave equation.
- The addition of the term $e\, A_\mu\, \dot{x}^\mu$ leads to relativistic charged point-particle dynamics, with magnetic moment ($g = 2$) and spin-orbit coupling.

GENERALIZING TO FLUIDS

- Consider the point-particle *à la* WONG again. Take a collection of particles indexed by λ.

$$S = -in \int dt \, \mathrm{Tr}(\sigma_3 \, g^{-1} \dot{g}) \quad \rightarrow \quad S = -i \int dt \sum_\lambda n_\lambda \mathrm{Tr}(\sigma_3 \, g_\lambda^{-1} \dot{g}_\lambda)$$

- We can take the continuum limit by $\lambda \rightarrow \vec{x}, \quad \sum_\lambda \rightarrow \int d^3x/v, \quad n_\lambda/v \rightarrow \rho(x)$.

- This leads to

$$S = -i \int d^4x \, \rho \, \mathrm{Tr}(\sigma_3 \, g^{-1} \dot{g})$$

  where $g = g(\vec{x}, t)$.

- This suggest the relativistic form

$$S = -i \int j^\mu \, \mathrm{Tr}(\sigma_3 \, g^{-1} \partial_\mu g)$$

- The difficulty for Poincaré is about what replaces $\dot{x}^\mu$. Only 3 of the 4 components are independent; further, role of diffeomorphisms versus translations in the Poincaré group is not clear.

THE LAGRANGIAN FOR ORDINARY FLUID DYNAMICS

- Ordinary fluid dynamics can be described by a Poisson bracket system

$$\begin{aligned} [\rho(x), \rho(y)] &= 0 \\ [v_i(x), \rho(y)] &= \partial_{xi}\delta^{(3)}(x-y) \\ [v_i(x), v_j(y)] &= -\frac{\omega_{ij}}{\rho}\,\delta^{(3)}(x-y) \end{aligned}$$

$\omega_{ij} = (\partial_i v_j - \partial_j v_i)$.

$$H = \int d^3x \left[\frac{1}{2}\rho\, v^2 \; + \; V(\rho)\right]$$

- We get the usual equations of fluid motion with pressure $p = \rho\frac{\partial V}{\partial \rho} - V$.
- The PBs can be summarized as

$$[F, G] = \int \left[\frac{\delta F}{\delta \rho}\partial_i\left(\frac{\delta G}{\delta v_i}\right) - \frac{\delta G}{\delta \rho}\partial_i\left(\frac{\delta F}{\delta v_i}\right) - \frac{\omega_{ij}}{\rho}\,\frac{\delta F}{\delta v_i}\,\frac{\delta G}{\delta v_j}\right]$$

for any two functions F, G.

THE LAGRANGIAN FOR ORDINARY FLUID DYNAMICS (*cont'd.*)

- The helicity C is given by

$$C = \frac{1}{8\pi} \int \epsilon^{ijk}\, v_i\, \partial_j v_k \;\; = \text{CS term for } v_i$$

- The helicity Poisson-commutes with all local observables, $[F, C] = 0$ for all F

  $\Longrightarrow C$ is superselected.

- Usually if $[\xi^a, \xi^b] = K^{ab}$, the Lagrangian is of the form $\mathcal{A}_a\, \dot{\xi}^a, \;\; \delta\mathcal{A} = \frac{1}{2}\, K^{-1}_{ab} \delta\xi^a \wedge \delta\xi^b$.

  Here K is not invertible, $\delta C / \delta v_i$ is a zero mode.

  This is the difficulty in writing down a Lagrangian.

- The solution is also clear: We must fix the value of C and seek a parametrization for the velocity which keeps the same value of C.

- Such a parametrization exists. It is the so-called Clebsch parametrization,

$$v_i = \partial_i \theta \;+\; \alpha\, \partial_i \beta$$

  θ, α, β are arbitrary functions.

THE LAGRANGIAN FOR ORDINARY FLUID DYNAMICS (*cont'd.*)

- For v_i parametrized in terms of well-defined θ, α, β,

$$C = \int (\text{total derivative}) = 0$$

- A suitable action which gives the PBs is now (C.C. LIN)

$$S = \int \rho\, \dot{\theta} + \rho\, \alpha\, \dot{\beta} - \int \left[\frac{1}{2} \rho\, v^2 + V \right]$$

- We can also write this as

$$S = \int J^\mu \; (\partial_\mu \theta + \alpha\, \partial_\mu \beta) \;\; - \;\; \int \left[J^0 \; - \; \frac{J^i J^i}{2\,\rho} + V \right]$$

$J^0 = \rho$; elimination of the auxiliary J^i leads to the previous version. $\int J^0$ is a constant.

- The relativistic generalization is

$$S = \int J^\mu \; (\partial_\mu \theta + \alpha\, \partial_\mu \beta) \;\; - \;\; \int F(n)$$

$F(n) = n + V(n), \qquad n^2 = J^2 = (J^0)^2 - J^i J^i.$

GROUP-THEORETIC DESCRIPTION OF FLUIDS

- The lesson from this is to treat
  - Translational part of action → Clebsch parametrization
  - Rest of the action in terms of the co-adjoint orbit version
- The general action is thus

$$S = \int d^4x \left[j^\mu \left(\partial_\mu \theta + \alpha \partial_\mu \beta \right) - \frac{i}{4} j^\mu_{(s)} \, \mathrm{Tr}(\Sigma_3 \, \Lambda^{-1} \partial_\mu \Lambda) + i \sum_a j^\mu_{(a)} \mathrm{Tr}(q_a \, g^{-1} D_\mu \, g) \right.$$
$$\left. - F(\{n\}) \right] \; + \; S(A)$$

- Generally, we must have different currents $j^\mu, j^\mu_{(s)}, j^\mu_{(a)}$ for mass flow, spin flow and the transport of other quantum numbers.
- Coupling to gauge fields follow from covariant derivatives on the group elements

GROUP-THEORETIC DESCRIPTION OF FLUIDS (*cont'd.*)

- $F(\{n\})$ depends on all invariant combinations of the currents and characterize the nature of the fluid, $n = \sqrt{j^\mu j_\mu}$, $n_a = \sqrt{j^\mu_{(a)} j_{\mu\,(a)}}$, etc.
- The group-valued fields are related to flow velocities and currents and given by the equations of motion,

$$\begin{aligned} \frac{1}{n}\frac{\partial F}{\partial n}\, j_\mu &= \partial_\mu \theta + \alpha\, \partial_\mu \beta \\ \frac{1}{n_a}\frac{\partial F}{\partial n_a}\, j_{\mu\,(a)} &= i\,\mathrm{Tr}\,(q_a\, g^{-1} D_\mu\, g), \qquad \text{etc.} \end{aligned}$$

Remark: The Clebsch parametrization can also be written in a "group" form,

$$-i\,\mathrm{Tr}(\sigma_3\, g^{-1} dg) = d\theta + \alpha\, d\beta$$

where $g \in SU(1,1)$ (or $SU(2)$),

$$g = \frac{1}{\sqrt{1-\bar{u}u}} \begin{pmatrix} 1 & u \\ \bar{u} & 1 \end{pmatrix} \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}, \qquad \alpha = \frac{\bar{u}u}{(1-\bar{u}u)}, \quad \beta = -\frac{i}{2}\log(u/\bar{u})$$

FURTHER COMMENTS

- In terms of the group-theoretic version, the helicity is given by the topological invariant

$$C = \text{constant} \int \mathrm{Tr}(g^{-1}\, dg)^3$$

- Another motivation for the action as we formulate it is:
  - The full quantum dynamics for a state with density matrix ρ is given by the action

$$S = \int dt\, \mathrm{Tr}\left[\rho_0 \left(U^\dagger i\, \frac{\partial U}{\partial t} - U^\dagger\, H\, U\right)\right]$$

  - The variational equation for this is

$$i\frac{\partial \rho}{\partial t} = H\,\rho - \rho\, H, \qquad \rho = U\,\rho_0\, U^\dagger$$

  - The canonical 1-form for this action is

$$\mathcal{A} = i\,\mathrm{Tr}(\rho_0\, U^\dagger\, \delta U)$$

  where δU includes all possible observables.

FURTHER COMMENTS (*cont'd.*)

- Consider a subset of transformations (symmetry transformations which can survive into the hydrodynamic regime),

$$\delta U = -i\,(t_A U)\,\delta\theta^A \;+\; \underbrace{\text{other transformations}}$$

neglect

- This corresponds to

$$\mathcal{A} = \mathrm{Tr}(U\,\rho_0\,U^\dagger\,t_A)\,\delta\theta^A = T_A\,\delta\theta^A, \qquad T_A(\theta) = \mathrm{Tr}\,(\rho\,t_A) = \langle t_A\rangle$$

  T_A will have appropriate group composition/commutation properties.

- θ's are essentially collective variables for the theory. The action (at the level of the θ's) which gives this $\mathcal{A}$ is the co-adjoint orbit action

FURTHER COMMENTS (*cont'd.*)

- Now a comment about the Clebsch variables:
  - Translational degrees of freedom x^μ can be used with the Poincaré group.
  - If we keep the $\dot{x}^\mu$ as fluid velocity, then we get the correct fluid equations, with no pressure.
  - To change to diffeomorphisms, look at the algebra

    $$[M(\xi), M(\xi')] = M(\xi \times \xi'), \qquad (\xi \times \xi')^i = \xi^k \partial_k {\xi'}^i - {\xi'}^k \partial_k \xi^i$$

    This can be realized by

    $$J_i = \pi_1 \partial_i \varphi_1 + \pi_2 \partial_i \varphi_2 + \cdots$$

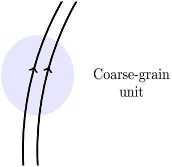
    for any number of pairs (π_i, φ_i).
  - We need two pairs for a complete characterization in 3 spatial dimensions.
  - Hence, we can argue

    Diffeomorphism symmetry $\Longrightarrow$ $SU(2)$ or $SU(1,1)$ symmetry

    for the pairs (π_i, φ_i), $\quad i = 1, 2$

FURTHER COMMENTS (*cont'd.*)

- π_1, φ_1 could be viewed as modulus and phase of ψ, ψ^*. How do we interpret the extra fields?
- For vorticity, we need to compare the velocities of nearby particles. Inside each coarse-graining unit (around, say, $\vec{x}$), we must have distinct fields representing these particles.
- $\psi(x)$ and $\psi(x+\epsilon)$ must be counted as independent fields since we want to replace them by fields at $\vec{x}$ upon coarse-graining.



WHAT IS TO FOLLOW

- We will discuss 3 examples
    - $SU(2)$ internal symmetry (Nonabelian Magnetohydrodynamics)
    - Magnetohydrodynamics including spin, magnetic moment and spin-orbit effects
    - Spin and coupling to gravity
- We will also discuss generalization to include anomalies
- We will also discuss applying the same formalism to braiding of vortices in a p-wave superconductor.

GENERAL ACTION FOR FLUIDS

- The general action is thus

$$S = \int d^4x \left[j^\mu \left(\partial_\mu \theta + \alpha \partial_\mu \beta\right) - \frac{i}{4} j^\mu_{(s)} \,\mathrm{Tr}(\Sigma_3 \Lambda^{-1} \partial_\mu \Lambda) + i \sum_a j^\mu_{(a)} \mathrm{Tr}(q_a g^{-1} D_\mu g) - F(\{n\})) \right] + S(A)$$

- Structure of action

Transport	Current	Fields
Mass flow	j^μ	θ, α, β
Spin flow	$j^\mu_{(s)}$	Lorentz group parameters Λ
Internal charge flow	$j^\mu_{(a)}$	Internal symmetry group element g

- Generally, we must have as many currents as the rank of the group, the corresponding densities are canonically conjugate to the diagonal angles.
- Any further relations among currents would "constitutive" relations, specific to the physical system.
- Coupling to gauge fields follow from covariant derivatives on the group elements

$SU(2)$ MAGNETOHYDRODYNAMICS

- Consider the action (BISTROVIC, JACKIW, LI, NAIR, PI)

$$S \;=\; \int J^\mu_m\,(\partial_\mu\theta + \alpha\,\partial_\mu\beta) - i\int j^\mu\,\mathrm{Tr}(\sigma_3\,g^{-1}D_\mu g) - \int F(n)\;+\;S_{YM}$$

$$\begin{aligned} D_\mu g &= \partial_\mu g + A_\mu\,g & A_\mu &= -i\,t^a\,A^a_\mu, \quad t^a = \tfrac{1}{2}\sigma^a\\ J^\mu_m &= n_m\,U^\mu, & U^2 &= 1\\ j^\mu &= n\,u^\mu, & u^2 &= 1 \end{aligned}$$

- We also include a background field which couples to the color charge.

$SU(2)$ MAGNETOHYDRODYNAMICS (*cont'd.*)

- The current which couples to A^a_μ is given by

$$J^{a\mu} = \mathrm{Tr}(\sigma_3\, g^{-1} t^a g)\, j^\mu = Q^a\, j^\mu, \qquad Q^a = \mathrm{Tr}(\sigma_3\, g^{-1} t^a g)$$

  This is the current for the nonabelian symmetry and has the Eckart form.

- Some of the equations of motion are

$$\begin{aligned} \partial_\mu j^\mu &= 0 \\ (D_\mu J^\mu)^a &= 0 \\ n\, u^\mu \partial_\mu (u_\nu F') - n\, \partial_\nu F' &= \mathrm{Tr}(J^\mu F_{\mu\nu}) \qquad (\text{“Euler equation”}) \end{aligned}$$

- The first two equations lead to the fluid generalization of the Wong equations

$$u^\mu (D_\mu Q)^a = (D_0 Q)^a + \vec{u} \cdot (\vec{D} Q)^a = 0$$

$SU(2)$ MAGNETOHYDRODYNAMICS (*cont'd.*)

- We also have

$$\partial_\mu T^{\mu\nu} = \mathrm{Tr}\,(J^\mu F_{\mu\nu})$$

  $T^{\mu\nu}$ has the perfect fluid form.

- The nonabelian charge density $\rho = \rho^a t^a$ (which is the time-component of $J^{a\mu}$) transforms, under gauge transformations, as

$$\rho \to \rho' = h^{-1} \rho\, h, \qquad\qquad h \in SU(2)$$

- We can diagonalize ρ at each point by an $(\vec{x}, t)$-dependent transformation, $\rho_{diag} = \rho_0 \sigma_3$. Then $\rho = g\, \rho_{diag}\, g^{-1}$, or

$$\rho^a = \rho_0\, \mathrm{Tr}(g\, \sigma_3\, g^{-1}\, t^a) = j^0\, \mathrm{Tr}(g\, \sigma_3\, g^{-1}\, t^a)$$

- g diagonalizes the charge density at each point. The eigenvalues are gauge-invariant and are represented by n. g describes the degrees of freedom corresponding to orientation in color space. Under a gauge transformation, $g \to h^{-1}\, g$.

$SU(2)$ MAGNETOHYDRODYNAMICS (*cont'd.*)

- There are two (related) charge densities, j^0 and the nonabelian charge density $\rho^a = J^{a\,0}$.
- The basic (new) Poisson brackets are

$$\begin{aligned}
\{j^0(\vec{x}), j^0(\vec{y})\} &= 0 \\
\{j^0(\vec{x}),\ g(\vec{y})\} &= -i\, g(\vec{x}) \left(\frac{\sigma_3}{2}\right) \delta(x-y) \\
\{\rho^a(\vec{x}), \rho^b(\vec{y})\} &= f^{abc} \rho^c(\vec{x})\, \delta(x-y) \\
\{\rho^a(\vec{x}),\ g(\vec{y})\} &= -i \left(\frac{\sigma_a}{2}\right) g(\vec{x})\, \delta(x-y)
\end{aligned}$$

- Remark: These equations of motion and charge algebra have some points of overlap with the work of GIBBONS, HOLM, KUPERSHMIDT
- A notable feature is (DAI, NAIR):

Since $\Pi_3[SU(N)] = \mathbb{Z}$, there are skyrmion-type (topological) solitons in any nonabelian magnetohydrodynamics

- Consider a special case where mass transport and charge transport are described by the same flow velocity.
- This applies when we have one species of particles with the same charge.
- Further, for dilute systems, if we neglect the possibility of spin-singlets forming (and moving independently), we can take spin flow velocity ≈ charge flow velocity
- The action for this case is (KARABALI, NAIR)

$$S = S(A) + \int d^4x \left[j^\mu \left(\partial_\mu \theta + \alpha \partial_\mu \beta + e A_\mu \right) - \frac{i}{4} j^\mu \, \mathrm{Tr}(\Sigma_3 \, \Lambda^{-1} \partial_\mu \Lambda) - F(n, \sigma) \right]$$

$\Lambda = B\, R$ contains the same velocity u^μ as in $j^\mu = n\, u^\mu$.

- F depends on n and $\sigma = S^{\mu\nu} F_{\mu\nu}$, where $S^{\mu\nu}$ is the spin density,

$$S^{\mu\nu} = \frac{1}{2} \, \mathrm{Tr}\, (\Sigma_3 \, \Lambda^{-1} \, J^{\mu\nu} \, \Lambda), \qquad J^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu]$$

ABELIAN MAGNETOHYDRODYNAMICS WITH SPIN (*cont'd.*)

- Having the same flow velocity leads to a requirement

$$\frac{2}{n}\frac{\partial F}{\partial n}\frac{\partial F}{\partial \sigma} = e$$

This is the fluid analog of the requirement of $g = 2$ for point-particles.

- The equations of motion are the Maxwell equations +

$$\begin{aligned}
u^{\alpha}\partial_{\alpha}(F' u_{\nu}) - \partial_{\nu}F' &= e\, u^{\lambda} F_{\lambda\nu} - \frac{16e}{F'}\partial_{\nu}S^{\lambda\beta}(S\,F\,S - F\,S\,S)_{\lambda\beta} + \cdots \\
u^{\alpha}\partial_{\alpha}S_{\mu\nu} &= \frac{e}{F'}\left[S_{\mu}^{\ \lambda}F_{\lambda\nu} - S_{\nu}^{\ \lambda}F_{\lambda\mu}\right] + \frac{1}{F'}\left[S_{\mu}^{\ \lambda}f_{\lambda\nu} - S_{\nu}^{\ \lambda}f_{\lambda\mu}\right] \\
&\quad - \frac{16\,e}{F'^{2}}(u_{\mu}S_{\nu}^{\ \lambda} - u_{\nu}S_{\mu}^{\ \lambda})\partial_{\lambda}S^{\rho\beta}(S\,F\,S - F\,S\,S)_{\rho\beta} + \cdots \\
f_{\lambda\nu} &= u_{\lambda}\,\partial_{\nu}F' - u_{\nu}\,\partial_{\lambda}F', \qquad F' = \frac{\partial F}{\partial n} \\
(S\,F\,S - F\,S\,S)_{\lambda\beta} &= S_{\lambda}^{\ \rho}\,F_{\rho\tau}\,S^{\tau}_{\ \beta} - F_{\lambda}^{\ \rho}\,S_{\rho\tau}\,S^{\tau}_{\ \beta}
\end{aligned}$$

Spin density is subject to precession effects due to pressure gradient terms as well as due to the external field.

INCORPORATING ANOMALIES

- First consider anomalous $U(1)$ symmetry, the fluid dynamical equations due to SON & SUROWKA.
- By use of the Clebsch parametrization, we can write the action (ABANOV, MONTEIRO, NAIR)

$$S = \int d^4x \left[j^\mu (V_\mu + A_\mu) + \frac{c}{6} \epsilon^{\mu\nu\alpha\beta} \left(A_\mu\, V_\nu \partial_\alpha V_\beta + V_\mu\, A_\nu \partial_\alpha A_\beta \right) - \mu \sqrt{-j^2} + P(\mu) \right]$$

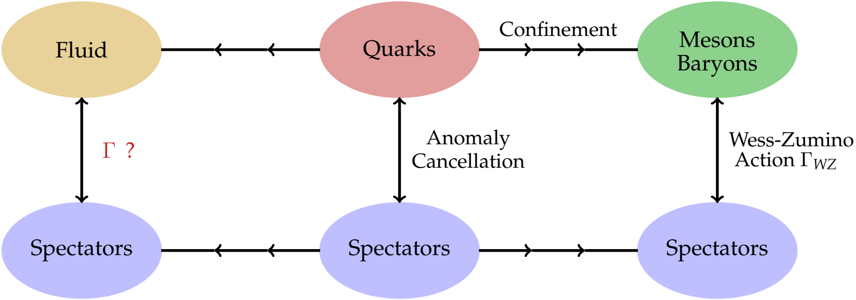
Here $V_\mu = \partial_\mu \theta + \alpha \partial_\mu \beta$.

- This leads to the anomaly equations

$$\begin{aligned} T^\mu_\nu &= \mu\, n\, U^\mu\, U_\nu + \delta^\mu_\nu\, P \\ J^\mu &= n\, U^\mu + \epsilon^{\mu\nu\alpha\beta} \left[\frac{c}{6}\, \mu U_\nu\, \partial_\alpha (\mu\, U_\beta) + \frac{c}{2} \mu U_\nu\, \partial_\alpha A_\beta \right] \\ \partial_\mu T^\mu_\nu &= F_{\lambda\mu}\, J^\mu, \qquad \partial_\mu J^\mu = -\frac{c}{8} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \\ (V + A)_\mu &= -\mu\, U_\mu \end{aligned}$$

Incorporating Anomalies (*cont'd.*)

- Now we turn to the standard model and the ′t Hooft argument for the Wess-Zumino action for anomalies



Fluid

Quarks

Confinement

Mesons
Baryons

Γ ?

Anomaly
Cancellation

Wess-Zumino
Action Γ_{WZ}

Spectators

Spectators

Spectators

- A similar argument for the fluid phase suggests an effective action for anomalies in terms of fluid variables. What is this action?

INCORPORATING ANOMALIES (*cont'd.*)

- Since we have formulated fluid dynamics using group variables, this is easy. We can use the same Γ_{WZ} but using fluid group element instead of meson fields.
- The suggestion is (NAIR, RAY, ROY)

$$
\begin{aligned}
S \;=\; & -i \int \Bigg[j_3^\mu \,\mathrm{Tr}\left(\frac{\lambda_3}{2}\, g_L^{-1} D_\mu\, g_L\right) + j_8^\mu \,\mathrm{Tr}\left(\frac{\lambda_8}{2}\, g_L^{-1} D_\mu\, g_L\right) + k_3^\mu \,\mathrm{Tr}\left(\frac{\lambda_3}{2}\, g_R^{-1} D_\mu\, g_R\right) \\
& + k_8^\mu \,\mathrm{Tr}\left(\frac{\lambda_8}{2}\, g_R^{-1} D_\mu\, g_R\right) + j_0^\mu \,\mathrm{Tr}\left(g_L^{-1} D_\mu\, g_L\right) + k_0^\mu \,\mathrm{Tr}\left(g_R^{-1} D_\mu\, g_R\right) \Bigg] \\
& - F(n_3, n_8, n_0, m_3, m_8, m_0) \;+\; S_{YM}(A) \;+\; \Gamma_{WZ}(A_L, A_R, g_L\, g_R^\dagger)
\end{aligned}
$$

- $\Gamma_{WZ}(A_L, A_R, g_L\, g_R^\dagger)$ is the standard Wess-Zumino term $\Gamma_{WZ}(A_L, A_R, U)$ with $U \Longrightarrow g_L\, g_R^\dagger$.
- There are other ways to incorporate anomalies (SON & SUROWKA; SADOFYEV & ISACHENKOV; ABANOV *et al*; WIEGMANN; + *many others*); an approach somewhat similar to ours is by SHU LIN.

INCORPORATING ANOMALIES (*cont'd.*)

- In full it is given by (WITTEN; KAYMAKCALAN, RAJEEV, SCHECHTER; + ...)

$$
\begin{aligned}
\Gamma_{WZ} = &- \frac{iN}{240\pi^2} \int_D \mathrm{Tr}(dU\, U^{-1})^5 - \frac{iN}{48\pi^2} \int_{\mathcal{M}} \mathrm{Tr}[(A_L\, dA_L + dA_L\, A_L + A_L^3)\, dUU^{-1}] \\
&- \frac{iN}{48\pi^2} \int_{\mathcal{M}} \mathrm{Tr}[(A_R\, dA_R + dA_R\, A_R + A_R^3)\, U^{-1}dU] \\
&+ \frac{iN}{96\pi^2} \int_{\mathcal{M}} \mathrm{Tr}[A_L\, dUU^{-1} A_L\, dUU^{-1} - A_R\, U^{-1}dU\, A_R\, U^{-1}dU] \\
&+ \frac{iN}{48\pi^2} \int_{\mathcal{M}} \mathrm{Tr}[A_L (dUU^{-1})^3 + A_R (U^{-1}dU)^3 + dA_L\, dU\, A_R\, U^{-1} - dA_R\, d(U^{-1})\, A_L\, U] \\
&+ \frac{iN}{48\pi^2} \int_{\mathcal{M}} \mathrm{Tr}[A_R\, U^{-1}\, A_L\, U (U^{-1}dU)^2 - A_L\, U\, A_R\, U^{-1} (dUU^{-1})^2] \\
&- \frac{iN}{48\pi^2} \int_{\mathcal{M}} \mathrm{Tr}[(dA_R\, A_R + A_R\, dA_R)\, U^{-1}\, A_L\, U - (dA_L\, A_L + A_L\, dA_L)\, U\, A_R\, U^{-1}] \\
&- \frac{iN}{48\pi^2} \int_{\mathcal{M}} \mathrm{Tr}[A_L\, U\, A_R\, U^{-1}\, A_L\, dUU^{-1} + A_R\, U^{-1}\, A_L\, U\, A_R\, U^{-1} dU] \\
&- \frac{iN}{48\pi^2} \int_{\mathcal{M}} \mathrm{Tr}[A_R^3\; U^{-1}\, A_L\, U - A_L^3\; U\, A_R\, U^{-1} + \tfrac{1}{2} U\, A_R\, U^{-1}\, A_L\, U\, A_R\, U^{-1}\, A_L]
\end{aligned}
$$

with $U \Longrightarrow g_L\, g_R^\dagger$.

- This action gives the chiral magnetic effect (& other anomaly related effects) for all flavor gauge fields and chemical potentials (A_0 components become the chemical potentials μ.)
- What is the chiral magnetic effect? (KHARZEEV, MCLERRAN, WARRINGA, FUKUSHIMA + ...)

In the quark-gluon plasma, in the presence of a magnetic field, because of the chiral anomaly

$\Longrightarrow$ Charge separation

$$J_0 = \frac{e^2}{2\pi^2}\,\nabla\theta \cdot \vec{B}$$

$\Longrightarrow$ Chiral induction

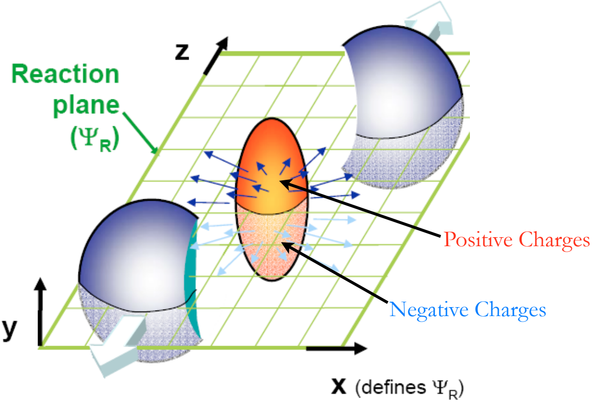
$$J_i = -\frac{e^2}{2\pi^2}\,\dot{\theta}\,B_i$$

- Here θ is like the θ-angle or η' field. In a plasma, $\dot{\theta} \rightarrow \frac{1}{2}(\mu_L - \mu_R)$, so

$$J_i = -\frac{e^2}{4\,\pi^2}\,(\mu_L - \mu_R)\,B_i$$

Chiral asymmetry leads to an electromagnetic current

ANOMALIES & CHIRAL MAGNETIC EFFECT (*cont'd.*)

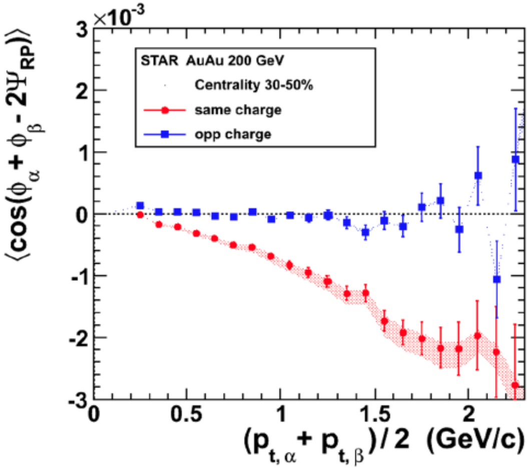


$$e\,B = \begin{cases} 10^{7}\,G & \textit{Highest in lab} \\ & \textit{before RHIC} \\ 10^{15}\,G & \textit{Magnetic stars} \\ 10^{17}\,G & \textit{RHIC} \end{cases}$$

- The electromagnetic current leads to charge separation which can be seen in asymmetry of charge distribution of final state particles.

ANOMALIES & CHIRAL MAGNETIC EFFECT (*cont'd.*)

- There is some experimental evidence for this.



(STAR *collaboration*)

ANOMALIES & CHIRAL MAGNETIC EFFECT (*cont'd.*)

- Going back to the WZ action, the electromagnetic current is given by (*previous refs, also* CALLAN & WITTEN)

$$\begin{aligned} J^\mu \quad &= \quad J_3^\mu + \frac{e}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\left[Q(\partial_\nu U\, U^{-1}\,\partial_\alpha U\, U^{-1}\,\partial_\beta U\, U^{-1}\right. \\ &\qquad \left. +U^{-1}\partial_\nu U\, U^{-1}\partial_\alpha U\, U^{-1}\partial_\beta U)\right] \\ &\qquad +i\frac{e^2}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}\partial_\nu A_\alpha \mathrm{Tr}\left[Q^2(\partial_\beta U\, U^{-1} + U^{-1}\partial_\beta U) + \frac{1}{2}(Q\partial_\beta U\, QU^{-1}\right. \\ &\qquad \left. -QUQ\partial_\beta U^{-1})\right] \end{aligned}$$

- We can restrict to two flavors by choosing

$$U = e^{i\theta}\begin{bmatrix} V & 0 \\ 0 & 1 \end{bmatrix}$$

ANOMALIES & CHIRAL MAGNETIC EFFECT (*cont'd.*)

- The current is now

$$
\begin{aligned}
J^\mu &= J_3^\mu + \frac{e}{48\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}(\mathcal{I}_\nu\,\mathcal{I}_\alpha\,\mathcal{I}_\beta) + i\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\,\partial_\nu A_\alpha\,\mathrm{Tr}\left[(\Sigma_{3L}+\Sigma_{3R})\,I_\beta\right] \; + J_\theta^\mu \\
J_\theta^\mu &= -\frac{e^2}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}\,\partial_\nu A_\alpha\,\partial_\beta\theta\,\left[2+\frac{1}{4}\mathrm{Tr}\,(\Sigma_{3L}\,\Sigma_{3R}-1)\right]
\end{aligned}
$$

$$\mathcal{I}_\beta = g_L^{-1}\partial_\beta g_L - g_R^{-1}\partial_\beta g_R, \qquad \Sigma_{3L} = g_L^{-1}\sigma_3 g_L,\ \Sigma_{3R} = g_R^{-1}\sigma_3 g_R.$$

- If we further restrict to $g_L = g_R$, we get

$$
\begin{aligned}
J_\theta^\mu &= -\frac{e^2}{2\pi^2}\epsilon^{\mu\nu\alpha\beta}(\partial_\nu A_\alpha)\,\partial_\beta\theta \\
J_i &= -\frac{e^2}{4\,\pi^2}\,(\mu_L-\mu_R)\,B_i
\end{aligned}
$$

  This reproduces the chiral magnetic effect which was originally calculated using Feynman diagrams (KHARZEEV, MCLERRAN, WARRINGA, FUKUSHIMA + ...).

- The full set of equations describe hydrodynamic transport of flavor charges.

THE CHIRAL ISOSPIN EFFECT

- The anomaly term Γ_{WZ} also has terms proportional to Z_μ, so there is also an induced isospin current (CAPASSO, NAIR, TEKEL).
- The relevant term is

$$\Gamma_{WZ} = -\frac{Ne^2}{6\,\pi^2}(\cot 2\theta_W) \int \epsilon^{\mu\nu\alpha\beta} Z_\mu \partial_\nu A_\alpha\, \partial_\beta \theta$$

- This leads to

$$\begin{aligned} J^{Z\,\mu} &= -\frac{e}{8\,\pi^2}(\cos 2\theta_W)\, \epsilon^{\mu\nu\alpha\beta} F_{\nu\alpha}\, \partial_\beta \theta \\ J^{3\,\mu} &= \frac{e}{8\,\pi^2}(\mu_L - \mu_R)\, B^i \end{aligned}$$

- In terms of pion fields, $J^{3\,\mu} \approx -\frac{1}{2} f_\pi \partial^\mu \Pi^0 + \cdots$. So we can interpret this as a pion field of gradient

$$\partial^i \Pi^0 = -\frac{e}{4\,\pi^2 f_\pi}(\mu_L - \mu_R)\, B^i$$

- This can manifest itself as an asymmetry in the neutral pion distribution.

ANOMALIES & CHIRAL VORTICITY EFFECT

- Generally, there is a contribution even when the background fields are zero.
- If we eliminate the group elements in favor of velocities, we get

$$\begin{aligned} J^\mu \quad &= \quad J_3^\mu + J_\theta^\mu + i\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\,\partial_\nu A_\alpha\,\mathrm{Tr}\left[(\Sigma_{3L}+\Sigma_{3R})\,I_\beta\right] \\ &\qquad + \frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\partial_\nu\,\mathrm{Tr}(g_L^{-1}\partial_\alpha g_L\; g_R^{-1}\partial_\beta g_R) \\ &\qquad + \frac{e}{12\pi^2}\epsilon^{\mu\nu\alpha\beta}\left[\left(\frac{\partial F}{\partial n_3}\right)^2 u_{3L\,\nu}\,\partial_\alpha u_{3L\,\beta} - \left(\frac{\partial F}{\partial m_3}\right)^2 u_{3R\,\nu}\,\partial_\alpha u_{3R\,\beta}\right]. \end{aligned}$$

A left-right asymmetry with nonzero vorticity can generate an electromagnetic current

- The standard model can have mixed gauge-gravity anomalies in some restricted cases. There are other anomaly related effects which can arise. We will not discuss them here (See notes and references).

Thank You