Isometry Group Orbit Quantization of Spinning Strings in $\mathrm{AdS}_3 \times \mathrm{S}^3$

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Contents

The AdS/CFT correspondence is one of the most fruitful research subject in the modern theoretical and mathematical physics.

It states the equivalence between $\mathcal{N} = 4$ supersymmetric Yang-Mills gauge theory in four-dimensional Minkowski space and superstring theory in the $AdS_5 \times S^5$ background.

Mapping the two sides to each other, weak coupling on one side is related to strong coupling on the other side. This makes the correspondence difficult to prove. But taken it for granted, one gains a very powerful tool for handling nonperturbative effects of string theory and gauge theory.

Another essential property of the correspondence is its holographic nature, i.e. a theory in a bulk is related to a theory on the boundary.

Up to now there exists no equivalence proof and the correspondence is realized via a set of mapping rules for certain classes of quantities describing the partner theories.

One of such rules is the map of conformal scaling dimensions of composite operators of the gauge theory to the energy spectrum of certain string configurations. Therefore, the computation of the string spectrum in $AdS \times S$ background became a classic problem in the AdS/CFT correspondence.

M.Heinze, G.J., L.Megrelidze [arXiv:1410.xxxx]

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AdS_3 and S^3 as group manifolds

AdS_3 is realized as a hyperboloid embedded in $\mathbb{R}^{2,2}$

$$(X^1)^2 + (X^2)^2 - (X^{0'})^2 - (X^0)^2 = -1$$

and S^3 is a sphere in $\mathbb{R}^4$

$$(Y^1)^2 + (Y^2)^2 + (Y^3)^2 + (Y^4)^2 = 1$$

The map to the group manifolds is given by

$$g = \begin{pmatrix} X^{0'} + iX^0 & X^1 + iX^2 \\ X^1 - iX^2 & X^{0'} - iX^0 \end{pmatrix} \qquad h = \begin{pmatrix} Y^4 + iY^3 & Y^2 + iY^1 \\ -Y^2 + iY^1 & Y^4 - iY^3 \end{pmatrix}$$

One finds: $g \in SU(1,1)$, $h \in SU(2)$. The length elements are related by

$$dX \cdot dX = \langle (g^{-1}\,dg)\,(g^{-1}\,dg) \rangle \qquad dY \cdot dY = \langle (h^{-1}\,dh)\,(h^{-1}\,dh) \rangle$$

Particle in $\mathrm{SU}(1,1)\times\mathrm{SU}(2)$

The dynamics of a particle in $SU(1,1)\times SU(2)$ is described by the action

$$S=\int \mathsf{d}\tau\;\left[\frac{\langle\, g^{-1}\dot{g}\, g^{-1}\dot{g}\,\rangle+\langle\, \tilde{g}^{-1}\dot{\tilde{g}}\, \tilde{g}^{-1}\dot{\tilde{g}}\,\rangle}{2\xi}-\frac{\xi\mu_{_0}^2}{2}\right]$$

Here τ is an evolution parameter, ξ plays the role of einbein and $\mu_{_0}$ is the particle mass. In the first order formalism, this action is equivalent to

$$S=\int \mathsf{d}\tau\;\left[\langle R\, g^{-1}\dot{g}\rangle+\langle\tilde{R}\,\tilde{g}^{-1}\dot{\tilde{g}}\rangle-\frac{\xi}{2}\Big(\langle RR\rangle+\langle\tilde{R}\tilde{R}\rangle+\mu_{_0}^2\Big)\right]$$

where R and $\tilde{R}$ are Lie algebra valued phase space variables, ξ becomes a Lagrange multiplier and its variation defines the mass-shell condition

$$\langle\, R\, R\,\rangle+\langle\,\tilde{R}\,\tilde{R}\,\rangle+\mu_{_0}^2=0$$

The Hamilton equations

$$g^{-1}\dot{g} = \xi R \qquad \tilde{g}^{-1}\dot{\tilde{g}} = \xi \tilde{R} \qquad \dot{R} = 0 \qquad \dot{\tilde{R}} = 0$$

provide the conservation of R and $\tilde{R}$, as well as of their 'left' counterparts

$$L = g\, R\, g^{-1} \qquad \tilde{L} = \tilde{g}\, \tilde{R}\, \tilde{g}^{-1}$$

The dynamical integrals L, $\tilde{L}$, R and $\tilde{R}$ are the Noether charges related to the invariance of the action with respect to the isometry transformations

$$g \mapsto g_l\, g\, g_r \ , \qquad \tilde{g} \mapsto \tilde{g}_l\, \tilde{g}\, \tilde{g}_r$$

Symplectic structure

The first order action defines the pre-symplectic form of the system

$$\Theta = \langle R g^{-1} \mathrm{d}g \rangle + \langle \tilde{R} \tilde{g}^{-1} \mathrm{d}\tilde{g} \rangle$$

which leads to the following Poisson brackets

$$\begin{array}{lll} \{L_\iota,\, L_\kappa\} = 2\epsilon_{\iota\kappa}{}^{\rho}\, L_\rho & \{R_\iota, R_\kappa\} = -2\epsilon_{\iota\kappa}{}^{\rho}\, R_\rho & \{L_\iota,\, R_\kappa\} = 0 \\ \{\tilde{L}_j,\, \tilde{L}_k\} = 2\epsilon_{jki}\, \tilde{L}_i & \{\tilde{R}_j,\, \tilde{R}_k\} = -2\epsilon_{jki}\, \tilde{R}_i & \{\tilde{L}_j,\, \tilde{R}_k\} = 0 \end{array}$$

where L_ι, L_j, R_ι, R_j are the components of the charges in some bases

$$L_\iota = \langle\, \mathfrak{b}_\iota\, L \,\rangle \qquad \tilde{L}_j = \langle\, \tilde{\mathfrak{b}}_j\, \tilde{L} \,\rangle \qquad R_\iota = \langle\, \mathfrak{b}_\iota\, R \,\rangle \qquad \tilde{R}_j = \langle\, \tilde{\mathfrak{b}}_j\, \tilde{R} \,\rangle$$

We use the following basis of the $\mathfrak{su}(1,1)$ algebra

$$\mathfrak{b}_0 = i\boldsymbol{\sigma}_3 \qquad \mathfrak{b}_1 = \boldsymbol{\sigma}_1 \qquad \mathfrak{b}_2 = \boldsymbol{\sigma}_2$$

A standard basis in $\mathfrak{su}(2)$ is given by $\tilde{\mathfrak{b}}_j = i\boldsymbol{\sigma}_j$ $(j = 1, 2, 3)$.

Gauge fixing

Since $R = R^\iota \mathfrak{b}_\iota$ and $\tilde{R} = \tilde{R}_j \tilde{\mathfrak{b}}_j$, the massshell condition can be written as

$$R_\iota R^\iota + \tilde{R}_j \tilde{R}_j + \mu_0^2 = 0$$

the components are gauge invariant and, therefore, the Poisson brackets algebra of the dynamical integrals will be preserved after a gauge fixing. Let us choose the gauge $\xi = 1$ and consider a solution in the $SU(1,1)$ part

$$g = e^{\mu\tau\mathfrak{b}_0} \qquad R = \mu\mathfrak{b}_0$$

which corresponds to the AdS_3 particle of mass μ in the rest frame. The isometry transformations provide a class of solutions parameterized by μ and the group variables

$$g = g_l \, e^{\mu\tau\mathfrak{b}_0} \, g_r \ , \qquad R = g_r^{-1} \mu\mathfrak{b}_0 \, g_r$$

Presymplectic form calculations

To find the Poisson bracket structure on the space of parameters, we calculate the $SU(1,1)$ part of the presymplectic form.
For fixed τ this calculation yields

$$\theta = \langle R g^{-1} \mathsf{d} g \rangle = \mu \langle\, \mathfrak{b}_0 \, g_l^{-1} \, \mathsf{d} g_l \,\rangle + \mu \langle\, \mathfrak{b}_0 \, \mathsf{d} g_r \, g_r^{-1} \,\rangle - \tau \mu \mathsf{d} \mu$$

and we can neglect the exact form $-\tau \mu \mathsf{d} \mu$.
With the help of the Cartan decomposition,

$$g_l = e^{\alpha_l \mathfrak{b}_0} \, e^{\gamma_l \mathfrak{b}_1} \, e^{\beta_l \mathfrak{b}_0} \qquad g_r = e^{\beta_r \mathfrak{b}_0} \, e^{\gamma_r \mathfrak{b}_1} \, e^{\alpha_r \mathfrak{b}_0}$$

one obtains a canonical 1-form

$$\theta = \mu \mathsf{d} \varphi + H_l \mathsf{d} \phi_l + H_r \mathsf{d} \phi_r$$

Presymplectic form calculations

The following notations are used

$$\varphi = -(\alpha_l + \beta_l + \alpha_r + \beta_r) \qquad \phi_l = \frac{\pi}{2} - 2\alpha_l \ , \quad \phi_r = \pi - 2\alpha_r$$

$$H_l = \frac{\mu}{2}\left[\cosh(2\gamma_l) - 1\right] \qquad H_r = \frac{\mu}{2}\left[\cosh(2\gamma_r) - 1\right]$$

The conserved Noether charges constructed from are given by

$$L = \mu\, e^{\alpha_l \mathfrak{b}_0}\, e^{\gamma_l \mathfrak{b}_1}\, \mathfrak{b}_0\, e^{-\gamma_l \mathfrak{b}_1}\, e^{-\alpha_l \mathfrak{b}_0} \qquad R = \mu\, e^{-\alpha_r \mathfrak{b}_0}\, e^{-\gamma_r \mathfrak{b}_1}\, \mathfrak{b}_0\, e^{\gamma_r \mathfrak{b}_1}\, e^{\alpha_r \mathfrak{b}_0}$$

and their components become

$$L^0 = \mu + 2H_l \qquad R^0 = \mu + 2H_r$$

$$L_\pm = \sqrt{\mu H_l + H_l^2}\, e^{\pm i\phi_l} \qquad R_\pm = \sqrt{\mu H_r + H_r^2}\, e^{\pm i\phi_r}$$

where $L_\pm = \frac{1}{2}(L_1 \pm i L_2)$ and $R_\pm = \frac{1}{2}(R_2 \pm i R_1)$.

Presymplectic form calculations

In the $SU(2)$ part we consider the isometry group orbit of the solution $\tilde{g} = e^{\tilde{\mu}\tau\tilde{\mathfrak{b}}_3}$.
Repeating here the same steps, we end up with a canonical 1-form

$$\tilde{\theta} \equiv \langle \tilde{R}\tilde{g}^{-1}\mathrm{d}\tilde{g} \rangle = \tilde{\mu}\mathsf{d}\tilde{\varphi} + \tilde{H}_l \mathsf{d}\tilde{\phi}_l + \tilde{H}_r \mathsf{d}\tilde{\phi}_r$$

The canonical coordinates parameterize the Noether charges $\tilde{L}_3$, $\tilde{L}_\pm = \frac{1}{2}(\tilde{L}_1 \pm i\tilde{L}_2)$ and $\tilde{R}_3$, $\tilde{R}_\pm = \frac{1}{2}(\tilde{R}_2 \pm i\tilde{R}_1)$ as follows

$$\tilde{L}_3 = \tilde{\mu} - 2\tilde{H}_l \qquad \tilde{R}_3 = \tilde{\mu} - 2\tilde{H}_r$$

$$\tilde{L}_\pm = \sqrt{\tilde{\mu}\tilde{H}_l - \tilde{H}_l^2}\; e^{\pm i\tilde{\phi}_l} \qquad \tilde{R}_\pm = \sqrt{\tilde{\mu}\tilde{H}_r - \tilde{H}_r^2}\; e^{\pm i\tilde{\phi}_r}$$

The Holstein-Primakoff realization

$$L^0 = \mu_a + 2a_L^\dagger a_L \qquad R^0 = \mu_a + 2a_R^\dagger a_R$$

$$L_+ = a_L^\dagger \sqrt{\mu_a + a_L^\dagger a_L} \qquad R_+ = a_R^\dagger \sqrt{\mu_a + a_R^\dagger a_R}$$

$$L_- = \sqrt{\mu_a + a_L^\dagger a_L}\; a_L \qquad R_- = \sqrt{\mu_a + a_R^\dagger a_R}\; a_R$$

$$L_3^s = \mu_s - 2a_L^{s\,\dagger} a_L^s \qquad R_3^s = \mu_s - 2a_R^{s\,\dagger} a_R^s$$

$$L_+^s = a_L^{s\,\dagger} \sqrt{\mu_s - a_L^{s\,\dagger} a_L^s} \qquad R_+^s = a_R^{s\,\dagger} \sqrt{\mu_s - a_R^{s\,\dagger} a_R^s}$$

$$L_-^s = \sqrt{\mu_s - a_L^{s\,\dagger} a_L^s}\; a_L^s \qquad R_-^s = \sqrt{\mu_s - a_R^{s\,\dagger} a_R^s}\; a_R^s$$

G.J, L.O'Raifeartaigh, I.Tsutsui, [hep-th/9407059].

$SU(1,1) \times SU(2)$ particle energy spectrum

The basis vectors of $su_l(1,1)) \oplus su_r(1,1)) \oplus su_l(2)) \oplus su_r(2)$ representation

$$| \, \mu_a; \, m_{_L} \, \rangle | \, \mu_a; \, m_{_R} \, \rangle | \, \mu_s; \, m^s_{_L} \, \rangle | \, \mu_s; \, m^s_{_R} \, \rangle$$

where $m_{_L}$, $m_{_R}$, μ_s, $m^s_{_L}$, $m^s_{_R}$ are nonnegative integers

$$m^s_{_L} \leq \mu_s \qquad m^s_{_R} \leq \mu_s$$

The representation is characterized by the Casimir numbers

$$C_A = -L_\mu L^\mu = -R_\mu R^\mu = \mu_a(\mu_a - 2)$$
$$C_S = \ L^s_m L^s_m = \ R^s_m R^s_m = \mu_s(\mu_s + 2)$$

The mass-shell condition $C_A = C_S + \mu^2$ defines μ_a by

$$\mu_a = 1 + \sqrt{\mu^2 + (\mu_s + 1)^2}$$

The spectrum of the energy operator

$$E = \mu_a + m_{_L} + m_{_R}$$

Spinning string in $SU(1,1) \times SU(2)$

The Polyakov action for the $SU(1,1) \times SU(2)$ string is given by

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int \mathsf{d}\tau \, \mathsf{d}\sigma \; \sqrt{-h} \, h^{\alpha\beta} \left(\langle \, g^{-1} \partial_\alpha g \, g^{-1} \partial_\beta g \, \rangle + \langle \, \tilde{g}^{-1} \partial_\alpha \tilde{g} \, \tilde{g}^{-1} \partial_\beta \tilde{g} \, \rangle \right)$$

where λ is a dimensionless coupling constant. Similarly to the particle dynamics, this action for a closed string is equivalent to

$$S = \int \mathsf{d}\tau \int_0^{2\pi} \frac{\mathsf{d}\sigma}{2\pi} \Big[\langle \mathcal{R} \, g^{-1} \dot{g} \rangle + \langle \tilde{\mathcal{R}} \, \tilde{g}^{-1} \dot{\tilde{g}} \rangle - \xi_2 \Big(\langle \mathcal{R} \, g^{-1} g' \rangle + \langle \tilde{\mathcal{R}} \, \tilde{g}^{-1} \tilde{g}' \rangle \Big) \\ - \frac{\xi_1}{2\sqrt{\lambda}} \Big(\langle \, \mathcal{R} \, \mathcal{R} \, \rangle + \langle \tilde{\mathcal{R}} \, \tilde{\mathcal{R}} \rangle + \lambda \langle (g^{-1} g')^2 \rangle + \lambda \langle (\tilde{g}^{-1} \tilde{g}')^2 \rangle \Big) \Big]$$

Hamiltonian description

The variations of Lagrange multipliers ξ_1 and ξ_2 provide the Virasoro constraints

$$\langle \mathcal{R}\,\mathcal{R} \rangle + \langle \tilde{\mathcal{R}}\,\tilde{\mathcal{R}} \rangle + \lambda \langle (g^{-1} g')^2 \rangle + \lambda \langle (\tilde{g}^{-1} \tilde{g}')^2 \rangle = 0$$

$$\langle \mathcal{R}\, g^{-1} g' \rangle + \langle \tilde{\mathcal{R}} \tilde{g}^{-1}\, \tilde{g}' \rangle = 0$$

The conformal gauge corresponds to $\xi_1 = 1$ and $\xi_2 = 0$.
In this case the Hamilton equations obtained from (1) become

$$\sqrt{\lambda}\; g^{-1} \dot{g} = \mathcal{R} \qquad \dot{\mathcal{R}} = \sqrt{\lambda} (g^{-1} g')'$$

$$\sqrt{\lambda}\; \tilde{g}^{-1} \dot{\tilde{g}} = \tilde{\mathcal{R}} \qquad \dot{\tilde{\mathcal{R}}} = \sqrt{\lambda} (\tilde{g}^{-1} \tilde{g}')'$$

and they are equivalent to

$$\partial_\tau \left(g^{-1} \dot{g}\right) = \partial_\sigma \left(g^{-1} g'\right) \qquad \partial_\tau \left(\tilde{g}^{-1} \dot{\tilde{g}}\right) = \partial_\sigma \left(\tilde{g}^{-1} \tilde{g}'\right)$$

Spinning string solutions

We consider the following solution of these equations

$$g = \begin{pmatrix} \cosh\vartheta\, e^{i(e\tau+m\sigma)} & \sinh\vartheta\, e^{i(p\tau+n\sigma)} \\ \sinh\vartheta\, e^{-i(p\tau+n\sigma)} & \cosh\vartheta\, e^{-i(e\tau+m\sigma)} \end{pmatrix}$$

$$\tilde{g} = \begin{pmatrix} \cos\tilde{\vartheta}\, e^{i(\tilde{e}\tau+\tilde{m}\sigma)} & i\sin\tilde{\vartheta}\, e^{i(\tilde{p}\tau+\tilde{n}\sigma)} \\ i\sin\tilde{\vartheta}\, e^{-i(\tilde{p}\tau+\tilde{n}\sigma)} & \cos\tilde{\vartheta}\, e^{-i(\tilde{e}\tau+\tilde{m}\sigma)} \end{pmatrix}$$

where the parameters are related by

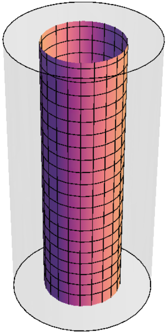
$$p^2 - e^2 = n^2 - m^2 \qquad \tilde{p}^2 - \tilde{e}^2 = \tilde{n}^2 - \tilde{m}^2$$

The Virasoro constraints lead to the additional conditions

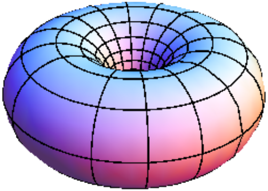
$$(e^2+m^2)\cosh^2\vartheta - (p^2+n^2)\sinh^2\vartheta = (\tilde{e}^2+\tilde{m}^2)\cos^2\tilde{\vartheta} + (\tilde{p}^2+\tilde{n}^2)\sin^2\tilde{\vartheta}$$

$$me\cosh^2\theta - np\sinh^2\vartheta = \tilde{m}\tilde{e}\cos^2\tilde{\vartheta} + \tilde{n}\tilde{p}\sin^2\tilde{\vartheta}$$

String worldsheets



(a)



(b)

The plot (a) corresponds to the AdS projection and the plot (b) to the spherical one.

Presymplectic form calculations

We analyze the presymplectic form

$$\Theta = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \left[\langle \mathcal{R}\, g^{-1} \mathrm{d} g \rangle + \langle \tilde{\mathcal{R}}\, \tilde{g}^{-1} \mathrm{d}\tilde{g} \rangle \right] .$$

To calculate this 1-form on the space of orbits one has to make the replacements

$$\mathcal{R} \mapsto \sqrt{\lambda}\, g_r^{-1}\, g^{-1} \dot{g}\, g_r \qquad g \mapsto g_l\, g\, g_r \qquad g^{-1} \mapsto g_r^{-1} g^{-1} g_l^{-1}$$

also similarly for the $SU(2)$ part, and after these replacements one has to identify g and $\tilde{g}$ with the solution.
This calculation yields

$$\Theta = \langle\, L\, g_l^{-1}\, \mathrm{d} g_l \,\rangle + \langle\, R\, \mathrm{d} g_r\, g_r^{-1} \,\rangle + \langle\, \tilde{L}\, \tilde{g}_l^{-1}\, \mathrm{d}\tilde{g}_l \,\rangle + \langle\, \tilde{R}\, \mathrm{d}\tilde{g}_r\, \tilde{g}_r^{-1} \,\rangle$$

Presymplectic form calculations

Here L, R, $\tilde{L}$ and $\tilde{R}$ are the Noether charges related to the isometry transformations

$$L = \sqrt{\lambda} \int_0^{2\pi} \frac{\mathsf{d}\sigma}{2\pi}\ \dot{g}\, g^{-1} \qquad R = \sqrt{\lambda} \int_0^{2\pi} \frac{\mathsf{d}\sigma}{2\pi}\ g^{-1} \dot{g}$$

$$\tilde{L} = \sqrt{\lambda} \int_0^{2\pi} \frac{\mathsf{d}\sigma}{2\pi}\ \dot{\tilde{g}}\, \tilde{g}^{-1} \qquad \tilde{R} = \sqrt{\lambda} \int_0^{2\pi} \frac{\mathsf{d}\sigma}{2\pi}\ \tilde{g}^{-1} \dot{\tilde{g}}$$

These charges are easily calculable and we obtain

$$L = \mu_l\, \mathfrak{b}_0 \qquad \tilde{L} = \tilde{\mu}_l\, \tilde{\mathfrak{b}}_3 \qquad R = \mu_r\, \mathfrak{b}_0 \qquad \tilde{R} = \tilde{\mu}_r\, \tilde{\mathfrak{b}}_3$$

where

$$\mu_l = \sqrt{\lambda}\, (e \cosh^2 \vartheta - p \sinh^2 \vartheta) \qquad \mu_r = \sqrt{\lambda}\, (e \cosh^2 \vartheta + p \sinh^2 \vartheta)$$

$$\tilde{\mu}_l = \sqrt{\lambda}\, (\tilde{e} \cos^2 \tilde{\vartheta} + \tilde{p} \sin^2 \tilde{\vartheta}) \qquad \tilde{\mu}_r = \sqrt{\lambda}\, (\tilde{e} \cos^2 \tilde{\vartheta} - \tilde{p} \sin^2 \tilde{\vartheta})$$

Presymplectic form calculations

The 1-form becomes similar to the case of particle

$$\Theta = \mu_l \langle\, \mathfrak{b}_0\, g_l^{-1}\, \mathsf{d}g_l \,\rangle + \mu_r \langle\, \mathfrak{b}_0\, \mathsf{d}g_r\, g_r^{-1} \,\rangle + \tilde{\mu}_l \langle\, \mathfrak{b}_3\, \tilde{g}_l^{-1}\, \mathsf{d}\tilde{g}_l \,\rangle + \tilde{\mu}_r \langle\, \mathfrak{b}_3\, \mathsf{d}\tilde{g}_r\, \tilde{g}_r^{-1} \,\rangle$$

and the same parametrization leads to the canonical 1-form

$$\Theta = \mu_l \mathsf{d}\varphi_l + H_l \mathsf{d}\phi_l + \mu_r \mathsf{d}\varphi_l + H_r \mathsf{d}\phi_r + \tilde{\mu}\mathsf{d}_l \tilde{\varphi}_l + \tilde{H}_l \mathsf{d}\tilde{\phi}_l + \tilde{\mu}\mathsf{d}_r \tilde{\varphi}_r + \tilde{H}_r \mathsf{d}\tilde{\phi}_r$$

The components of the symmetry generators have the same form

$$L^0 = \mu_l + 2H_l \qquad R^0 = \mu_r + 2H_r$$

$$L_\pm = \sqrt{\mu_l H_l + H_l^2}\, e^{\pm i\phi_l} \qquad R_\pm = \sqrt{\mu_r H_r + H_r^2}\, e^{\pm i\phi_r}$$

$$\tilde{L}_3 = \tilde{\mu}_l - 2\tilde{H}_l \qquad \tilde{R}_3 = \tilde{\mu}_r - 2\tilde{H}_r$$

$$\tilde{L}_\pm = \sqrt{\tilde{\mu}_l \tilde{H}_l - \tilde{H}_l^2}\; e^{\pm i\tilde{\phi}_l} \qquad \tilde{R}_\pm = \sqrt{\tilde{\mu}_r \tilde{H}_r - \tilde{H}_r^2}\; e^{\pm i\tilde{\phi}_r}$$

Here, now the Casimir numbers μ_l and μ_r are independent, whereas $\tilde{\mu}_l$ and $\tilde{\mu}_r$ are integers of the same parity, which ensures that the total angular momentum $\frac{1}{2}(\tilde{\mu}_l + \tilde{\mu}_l)$ on S^3 takes integer values.

The energy spectrum

The Holstein-Primakoff transformation here provides a similar realization of the isometry group generators with non-negative integers $\tilde{\mu}_l$, $\tilde{\mu}_r$.
The energy spectrum becomes

$$E = E_0 + m_l + m_r$$

where m_l and m_r are non-negative integers and $E_0 = \sqrt{\lambda}\, e \cosh^2 \vartheta$ corresponds to the minimal energy for a given $\tilde{\mu}_l$, $\tilde{\mu}_r$.

The dependence of E_0 on $\tilde{\mu}_l$, $\tilde{\mu}_r$ and the winding numbers is obtained from the constraints.
One gets a fourth order equation for $\cos 2\tilde{\vartheta}$

$$(\tilde{\mu}_l + \tilde{\mu}_r)^2 (1 - \cos 2\tilde{\vartheta})^2 - (\tilde{\mu}_l - \tilde{\mu}_r)^2 (1 + \cos 2\tilde{\vartheta})^2 = \lambda(\tilde{m}^2 - \tilde{n}^2)(1 - \cos^2 2\tilde{\vartheta})^2$$

Solutions of this equation define E_0 through the Virasoro constraints.

The energy spectrum

The solutions of at large λ for $\tilde{n} = 0$ are

$$\cos 2\tilde{\vartheta} = -1 + \frac{|\tilde{\mu}_l + \tilde{\mu}_r|}{|\tilde{m}|}\,\lambda^{-1/2} + \mathcal{O}(\lambda^{-1}) \qquad \tilde{p} = \frac{\tilde{\mu}_l - \tilde{\mu}_r}{2}\lambda^{-1/2} + \mathcal{O}(\lambda^{-1})$$

which leads to

$$\sinh^2 \vartheta = \frac{|\tilde{m}(\tilde{\mu}_l + \tilde{\mu}_r)|}{2n^2}\lambda^{-1/2} + \mathcal{O}(\lambda^{-1}) \quad e^2 = 2|\tilde{m}(\tilde{\mu}_l + \tilde{\mu}_r)|\lambda^{-1/2} + \mathcal{O}(\lambda$$

$$E_0 = \sqrt{2|\tilde{m}(\tilde{\mu}_l + \tilde{\mu}_r)|}\,\lambda^{1/4} + \mathcal{O}(\lambda^{-1/4})$$

The case $\mu_l = \mu_r$ is special and gives the simple solution in the $SU(2)$ part

$$\tilde{p} = 0, \qquad \tilde{e} = |\tilde{m}| \qquad \cos^2 \tilde{\vartheta} = \frac{\tilde{\mu}_l}{|\tilde{m}|}\,\lambda^{-1/2}$$

and the corresponding minimal energy

$$E_0 = 2\sqrt{|\tilde{m}\tilde{\mu}_l|}\,\lambda^{1/4} + \mathcal{O}(\lambda^{-1/4})$$

Summary and outlook

Particle and string dynamics in $\mathrm{AdS}_3 \times \mathrm{S}^3$ is described in terms of $\mathrm{SU}(1,1) \times \mathrm{SU}(2)$ group variables.

Hamiltonian treatment of the isometry group orbits of solutions in a fixed gauge is obtained via the analysis of the presymplectic form.

The space of solutions of particle dynamics is covered by a one-parameter family of the orbits parameterized by creation-annihilation type variables leading to the Holstein-Primakoff realization of the isometry group generators.

The scheme is then applied to two-parameter family of spinning string solutions with winding numbers.

It again provides an oscillator type realization of the symmetry generators, however, with more freedom in the Casimir numbers.

The energy has an ascillator type spectrum. Analyzing the minimal energy at strong coupling we verify the spectrum of short strings at strong coupling.

The aim is to extend this description to supergroup orbits.